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Simulation of mass flows of decaying substance in layer with periodically located thin channels

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In this paper the processes of admixture convective diffusion in two-phase structures with periodically located thin channels are investigated with taking into account a natural decay of migrating substance. With the help of application of appropriate integral transforms separately in the contacting domains, a solution of the contact initial boundary value problem of convective diffusion of decaying substance is obtained. The correlations between these integral transforms are found using the non-ideal contact conditions imposed for the concentration function. Expressions for decaying particle flows through arbitrary cross-section of the body are found and investigated, and their numerical analysis is carried out in the middle of both domains — the thin channel and basic material. It is shown that the decay intensity of the migrating substance especially affects the flow distribution in the domain of basic material.

Keywords: *diffusion, convection, admixture decay, regular structure, thin channel, mass flow*

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1. Introduction

Forecasting the processes that occur in our environment, many engineering calculations need a mathematical description of the processes of diffusion, filtration, etc. in complex media, including space-regular ones. Such media include, for example, the concrete of linked pores structure, soils, which can be regarded as a regular two-phase structure containing periodically located thin channels in which the transfer of particles going on by both diffusive and convective mechanisms. Herewith, the migrating admixture can decay as a result of either chemical reactions or radioactive decay. Since the construction of exact solutions of this type of problems even for simple geometric areas cause difficulties, there usually used approximate analytic [1,2] or numerical [3,4] solutions.

To solve the problems of diffusion in such media, a method has been suggested, which is based on the use of integral transforms for the spatial variables applied separately in the contacting domains [5]. In [6,7], this method has been generalized to the case when in sublayers of the one type of periodic structure both diffusive and convective transports are taken into account, while in sublayers of the other type, only the diffusion mechanism of mass transfer is allowed for. Boundary conditions of the first kind on the concentration which takes different values on the "top" surface of different structural elements of the body are considered as well as mixed boundary conditions.

In this paper, for a two-phase layer of regular structure, the non-stationary cases of admixture diffusion processes with taking into account both the convective mass transfer mechanism in one of the phases and the decay of the migrating admixture are investigated. Expressions for the concentrations of decaying substance are written down as well as expressions for the diffusion flow through arbitrary cross-section of the body are obtained, a numerical analysis is carried out.

2. Subject of inquiry and formulation of the problem

Let decaying substance of one chemical type is migrating in a layer of the thickness x_0, which consists of periodically located phases of two kinds. The surfaces separating these phases are normal to the layer boundaries (Fig. 1, a) (the Ox axis is normal to the body surface; the Oy axis is normal to the lateral phase boundaries). The phases whose diffusion coefficient is D_1 have the width $2L$ and the phases whose diffusion coefficient is D_2 have the width $2l$; herewith, in the domains with the diffusion coefficient D_1, the convective mass transport is taken into account with the coefficient of convective velocity v, which is known and constant. This structure has a family of planes of symmetry ($y = \pm n(L + l)$, $n = 0, 1, 2, \ldots$) which divide the contacting phases into two equal parts. Therefore, we can conventionally separate off an element of the body (Fig. 1, b) on whose vertical surfaces the mass flow in the direction of the Oy axis is equal to zero.

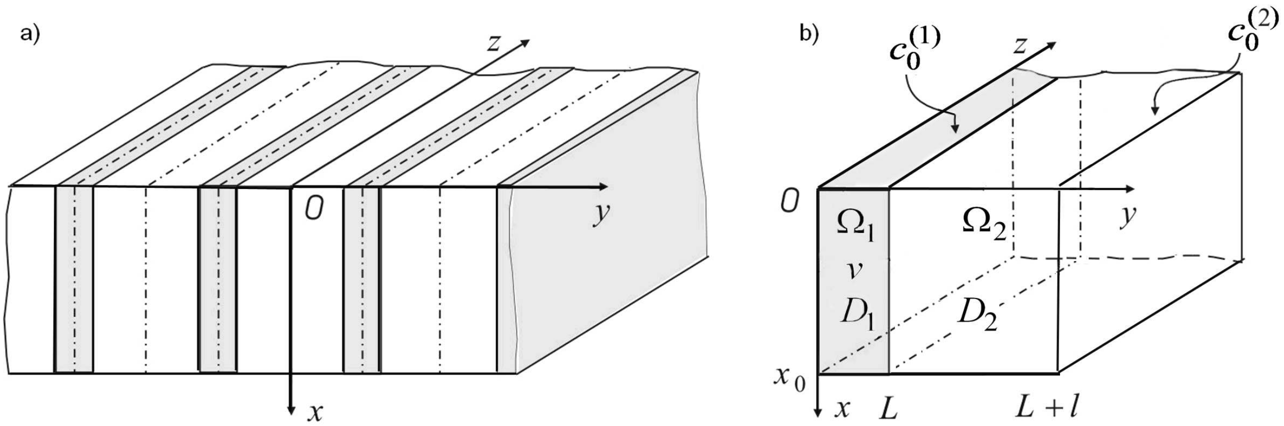

Fig. 1. Regular structure of the body, where admixture substance is migrating (a); a chosen element of such structure (b).

In the nonstationary case, the concentration of decaying admixture $c_1(x, y, t)$ in the domain $\Omega_1 =]0; x_0[\times]0; L[$ is defined from the equation of convective diffusion:

$$\frac{\partial c_1}{\partial t} = D_1 \left[\frac{\partial^2 c_1}{\partial x^2} + \frac{\partial^2 c_1}{\partial y^2}\right] - v\frac{\partial c_1}{\partial x} - \lambda c_1, \quad x, y \in \Omega_1, \tag{1}$$

where λ is the coefficient of decay intensity of the migrating substance [c^{-1}].

In the domain $\Omega_2 = [0; x_0] \times [L; L + l]$ the concentration of decaying admixture particles $c_2(x, y, t)$ satisfies the diffusion equation

$$\frac{\partial c_2}{\partial t} = D_2 \left[\frac{\partial^2 c_2}{\partial x^2} + \frac{\partial^2 c_2}{\partial y^2}\right] - \lambda c_2, \quad x, y \in \Omega_2. \tag{2}$$

In the initial time, we assume the admixture in the body is absent:

$$c_1(x, y, t)|_{t=0} = c_2(x, y, t)|_{t=0} = 0. \tag{3}$$

At the upper surface of the layer $x = 0$, the admixture concentration values are maintained to be constant and at the lower surface the concentrations are equal to zero:

$$c_1(x, y, t)|_{x=0} = c_0^{(1)} \equiv \text{const}, \quad c_2(x, y, t)|_{x=0} = c_0^{(2)} \equiv \text{const};$$

$$c_1(x, y, t)|_{x=x_0} = c_2(x, y, t)|_{x=x_0} = 0. \tag{4}$$

At the lateral surfaces of the chosen element $y = 0$, $y = L + l$, the horizontal components of the flow are equal to zero, namely

$$\left.\frac{\partial c_1(x,y,t)}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial c_2(x,y,t)}{\partial y}\right|_{y=L+l} = 0. \tag{5}$$

On the contact interface $y = L$, we impose the condition of nonideal contact for the concentration function in the form [8]:

$$\eta_1 c_1(x,y,t)\left.\right|_{y=L} = \eta_2 c_2(x,y,t)\left.\right|_{y=L}, \quad D_1 \left.\frac{\partial c_1(x,y,t)}{\partial y}\right|_{y=L} = D_2 \left.\frac{\partial c_2(x,y,t)}{\partial y}\right|_{y=L}, \tag{6}$$

where η_1 and η_2 ($\eta_1 \neq \eta_2$) are the coefficients of the concentration dependence of the chemical potentials in each domain.

3. Construction of the solution of the formulated problem

We seek a solution of the contact initial boundary value problem (1)–(6) of the mass transfer with the use of integral transforms for spatial variables separately in the contacting domains [5–8]. In order to do this, we have denoted the function $\partial c_1/\partial y$ on the boundary of the domain Ω_1 and $\partial c_2/\partial y$ on the boundary of Ω_2, taking into account the second contact condition (6), which means that at the interface $y = L$ the mass flows are equal to each other and, in their turn, are equal to some function $g(x,t)$, i.e.

$$D_1 \left.\frac{\partial c_1(x,y,t)}{\partial y}\right|_{y=L} = D_2 \left.\frac{\partial c_2(x,y,t)}{\partial y}\right|_{y=L} = g(x,t). \tag{7}$$

Then, we can perform finite integral cos-transform with respect to the variable y in the domain Ω_1 [10]: $y \to y_k = k\pi/L$, $c_1(x,y,t) \to \bar{c}_1(x,k,t)$, and the cos-transform with a shift in the domain Ω_2 [11]: $y \to y_j = j\pi/l$, $c_2(x,y,t) \to \bar{c}_2(x,j,t)$. With respect to the variable x, in the domain Ω_1 we apply the following integral transform [12]

$$\bar{c}_1(n,k,t) = \int_0^{x_0} \tilde{c}_1(x,k,t) e^{-v_D x} \sin(x_n x) dx, \tag{8}$$

$$\tilde{c}_1(x,k,t) = \frac{2}{x_0} e^{v_D x} \sum_{n=1}^{\infty} \bar{c}_1(n,k,t) \sin(x_n x), \tag{9}$$

where $v_D = v/2D_1$, $x_n = n\pi/x_0$, and in the domain Ω_2 we apply finite integral Fourier sin-transform [10]: $x \to x_m = m\pi/x_0$, $\tilde{c}_2(x,j,t) \to \bar{c}_2(m,j,t)$.

As a result, we obtain a solution of the problem (1)–(6) in the form [13]:

$$\bar{c}_1(n,k,t) = e^{-\left[D_1\left(x_n^2+y_k^2+v_D^2\right)+\lambda\right]t} \int_0^t \left[D_1 a_k c_0^{(1)} x_n + (-1)^k \tilde{g}_n(t')\right] e^{\left[D_1\left(x_n^2+y_k^2+v_D^2\right)+\lambda\right]t'} dt'. \tag{10}$$

$$\bar{c}_2(m,j,t) = e^{-\left[D_2\left(x_m^2+y_j^2\right)+\lambda\right]t} \int_0^t \left[D_2 a_j c_0^{(2)} x_m - \tilde{g}_m(t')\right] e^{\left[D_2\left(x_m^2+y_j^2\right)+\lambda\right]t'} dt'. \tag{11}$$

Note, that for the function $g(x,t)$ we have relatively

$$\tilde{g}_n(t) = \int_0^{x_0} g(x,t) e^{-v_D x} \sin(x_n x) dx, \quad \tilde{g}_m(t) = \int_0^{x_0} g(x,t) \sin(x_m x) dx. \tag{12}$$

Remark that inverse integral transforms to (12) are the following

$$g(x,t) = e^{v_D x} \sum_{n=1}^{\infty} \tilde{g}_n(t) \sin(x_n x), \quad g(x,t) = \frac{2}{x_0} \sum_{m=1}^{\infty} \tilde{g}_m \sin(x_m x) \; . \tag{13}$$

In the expressions (10) and (11), the functions $\tilde{g}_n(t')$ and $\tilde{g}_m(t')$ remain still undetermined. We seek these functions using both the first contact condition (6) of the jump of the concentration function at the interface of the domains Ω_1 and Ω_2 and the transform (13). To do this, perform the corresponding inverse integral transforms and substitute the obtained expressions into the first condition (6). As a result, we get the following integral equation:

$$\begin{aligned} \int_0^t \eta_1 e^{v_D x} \sum_{n=1}^{\infty} \sin(x_n x) \Bigg(\left[D_1 c_0^{(1)} x_n + \frac{\tilde{g}_n(t')}{L} \right] e^{-\left[D_1\left(v_D^2 + x_n^2\right) + \lambda\right](t-t')} + \\ + \frac{2}{L} \tilde{g}_n(t') \sum_{k=1}^{\infty} (-1)^k e^{-\left[D_1\left(v_D^2 + x_n^2 + y_k^2\right) + \lambda\right](t-t')} \Bigg) dt' = \\ = \int_0^t \eta_2 \sum_{m=1}^{\infty} \sin(x_m x) \Bigg(\left[D_2 c_0^{(2)} x_m - \frac{\tilde{g}_m(t')}{l} \right] e^{-\left[D_2 x_m^2 + \lambda\right](t-t')} - \\ - \frac{2}{l} \tilde{g}_m(t') \sum_{j=1}^{\infty} e^{-\left[D_2\left(x_m^2 + y_j^2\right) + \lambda\right](t-t')} \Bigg) dt'. \end{aligned} \tag{14}$$

Beside this, we need to find the correlation between the functions $\tilde{g}_n(t')$ and $\tilde{g}_m(t')$. It appeared that taking into account the decay of migrating substance does not influence this correlation. Thus we have [7]

$$\tilde{g}_m(t') = \frac{2}{x_0} \sum_{n=1}^{\infty} A_{n,m} \tilde{g}_n(t') \quad \text{or} \quad \tilde{g}_n(t') = \frac{2}{x_0} \sum_{m=1}^{\infty} B_{n,m} \tilde{g}_m(t'), \tag{15}$$

where the coefficients $A_{n,m}$ and $B_{n,m}$ are determined in the following way

$$A_{n,m} \equiv \frac{2 v_D \pi^2}{x_0^2} \frac{nm\left[(-1)^{n+m} e^{v_D x_0} - 1\right]}{\left\{ v_D^2 + \frac{\pi^2}{x_0^2}(n-m)^2 \right\} \left\{ v_D^2 + \frac{\pi^2}{x_0^2}(n+m)^2 \right\}},$$

$$B_{n,m} \equiv -\frac{2 v_D \pi^2}{x_0^2} \frac{nm\left[(-1)^{n+m} e^{-v_D x_0} - 1\right]}{\left\{ v_D^2 + \frac{\pi^2}{x_0^2}(n-m)^2 \right\} \left\{ v_D^2 + \frac{\pi^2}{x_0^2}(n+m)^2 \right\}}.$$

Having solved the equation (14), we obtain

$$\tilde{g}_n(t') = \frac{ -\frac{\eta_1}{\eta_2} D_1 c_0^{(1)} x_n + \frac{2}{x_0} D_2 c_0^{(2)} \sum\limits_{m=1}^{\infty} x_m B_{n,m} E_{n,m}(t-t') }{ \frac{\eta_1}{\eta_2 L} \Theta_0 \left(0, e^{-D_1 \frac{\pi^2}{L^2}(t-t')} \right) + \frac{4}{x_0^2 l} \sum\limits_{m=1}^{\infty} \Phi_m(t-t') A_{n,m} B_{n,m} e^{D_1\left(v_D^2 + x_n^2\right)(t-t')} }, \tag{16}$$

where $E_{n,m}(t-t') = \exp\left\{ -\left[D_2 x_m^2 - D_1\left(v_D^2 + x_n^2\right)\right](t-t')\right\}$, $\Theta_0(v,x)$ is the elliptic theta function [14], $\Phi_m(t-t') = e^{-D_2 x_m^2 (t-t')} \left\{ 1 + 2 \sum\limits_{j=1}^{\infty} e^{-D_2 y_j^2 (t-t')} \right\}$.

Finally we obtain equations for the concentration of decaying admixture substance in the domain Ω_1

$$c_1(x,y,t) = e^{-\lambda t} e^{v_D x}\left[c_0^{(1)}\frac{\sinh v_D\,(x_0-x)}{\sinh\,(v_D x_0)} + \frac{2}{x_0}\sum_{n=1}^{\infty}\sin\,(x_n x)\left[-\frac{c_0^{(1)}x_n}{v_D^2+x_n^2}e^{-D_1\left(v_D^2+x_n^2\right)t} + \right.\right.$$
$$\left.\left. + \frac{1}{L}\int\limits_0^t\left\{\tilde{g}_n(t')e^{\lambda t'}e^{-D_1\left(v_D^2+x_n^2\right)(t-t')}\left(1+2\sum_{k=1}^{\infty}(-1)^k\cos\,(y_k y)\,e^{-D_1 y_k^2(t-t')}\right)\right\}dt'\right]\right], \quad (17)$$

and in the domain Ω_2

$$c_2(x,y,t) = c_0^{(2)}\left(1-\frac{x}{x_0}\right) - \frac{2}{x_0}\sum_{m=1}^{\infty}\sin\,(x_m x)\left[\frac{c_0^{(2)}e^{-\lambda t}e^{-D_2 x_m^2 t}}{x_m} + \right.$$
$$\left. + \frac{1}{l}\int\limits_0^t \tilde{g}_m(t')e^{-\left[D_2 x_m^2+\lambda\right](t-t')}\left(1+2\sum_{j=1}^{\infty}(-1)^j\cos\,(y_j\,(y-L))\,e^{-D_2 y_j^2(t-t')}\right)dt'\right]. \quad (18)$$

Note, that in order to determine $\tilde{g}_m(t')$, we have used the correlation (15) with the expression (16).

4. Mass flows of the decaying substance

Obtained analytical solutions of the contact initial boundary value problem of the convective diffusion in regular structures under boundary conditions of the first kind make it possible to find nonstationary mass flows of decaying particles for this problem trough arbitrary surfaces $x = x_*$ and $y = y_*$, which are determined by the formulae [7]
in the domain Ω_1

$$J_{*x}^{(1)}(y,t) = -D_1\frac{\partial c_1(x,y,t)}{\partial x} + v c_1(x,y,t)\bigg|_{x=x_*}; \quad J_{*y}^{(1)}(x,t) = -\,D_1\frac{\partial c_1(x,y,t)}{\partial y}\bigg|_{y=y_*}; \quad (19)$$

in the domain Ω_2

$$J_{*x}^{(2)}(y,t) = -\,D_2\frac{\partial c_2(x,y,t)}{\partial x}\bigg|_{x=x_*}; \quad J_{*y}^{(2)}(x,t) = -\,D_2\frac{\partial c_2(x,y,t)}{\partial y}\bigg|_{y=y_*}. \quad (20)$$

Substitute the corresponding expressions (17) and (18) for the concentrations $c_i(x,y,t)$ into the relations (19), (20). Then we obtain
in the domain Ω_1
the mass flow $J_{*x}^{(1)}(y,t)$ through the surface $x = x_*$

$$J_{*x}^{(1)}(y,t) = -\,D_1\frac{\partial c_1}{\partial x} + v c_1\bigg|_{x=x_*} = e^{-\lambda t}e^{v_D x_*}\left[(v-D_1 v_D)c_0^{(1)}\frac{\sinh v_D\,(x_0-x)}{\sinh\,(v_D x_0)} + \right.$$
$$+ D_1 v_D c_0^{(1)}\frac{\cos v_D(x_0-x)}{\sinh v_D x_0} + \frac{2}{x_0}\sum_{n=1}^{\infty}\left\{\,(v-D_1\,v_D)\sin\,(x_n x_*) - D_1 x_n\cos(x_n x_*)\times\right.$$
$$\times\left[-\frac{c_0^{(1)}x_n}{v_D^2+x_n^2}e^{-D_1\left(v_D^2+x_n^2\right)t} + \frac{1}{L}\int\limits_0^t\left\{\tilde{g}_n(t')e^{\lambda t'}e^{-D_1\left(v_D^2+x_n^2\right)(t-t')}\right\}\times\right.$$

$$\times \left(1+2\sum_{k=1}^{\infty}(-1)^k \cos\left(y_k y\right)\ e^{-D_1 y_k^2 (t-t')}\right) dt' \Bigg] \Bigg\} \Bigg];$$

in particular, the mass flow $J_{0x}^{(1)}(y,t)$ through the lower surface $x = x_0$

$$J_{0x}^{(1)}(y,t) = e^{-\lambda t} e^{v_D x_0} \left[\frac{D_1 v_D c_0^{(1)}}{\sinh\left(v_D x_0\right)} - \frac{2}{x_0} D_1 \sum_{n=1}^{\infty} (-1)^{n+1} x_n \times \right.$$
$$\times \left[-\frac{c_0^{(1)} x_n}{v_D^2 + x_n^2} e^{-D_1\left(v_D^2+x_n^2\right)t} + \frac{1}{L}\int_0^t \left\{ \tilde{g}_n(t') e^{\lambda t'} e^{-D_1\left(v_D^2+x_n^2\right)(t-t')} \right\} \times \right.$$
$$\left.\left. \times \left(1+2\sum_{k=1}^{\infty}(-1)^k \cos\left(y_k y\right)\ e^{-D_1 y_k^2 (t-t')}\right) dt' \right] \right];$$

the mass flow $J_{*y}^{(1)}(x,t)$ through the surface $y = y_*$

$$J_{*y}^{(1)}(x,t) = -D_1 \left.\frac{\partial c_1}{\partial y}\right|_{y=y_*} = e^{-\lambda t} e^{v_D x} \frac{4 D_1}{L x_0} \sum_{n=1}^{\infty} \sin(x_n x) \times$$
$$\times \int_0^t \left\{ \tilde{g}_n(t') e^{\lambda t'} e^{-D_1\left(v_D^2+x_n^2\right)(t-t')} \left(1+2\sum_{k=1}^{\infty}(-1)^k \cos\left(y_k y\right) e^{-D_1 y_k^2 (t-t')}\right) \right\} dt'. \qquad (21)$$

In the domain Ω_2:
the mass flow $J_{*x}^{(2)}(y,t)$ through the surface $x = x_*$

$$J_{*x}^{(2)}(y,t) = -\, D_2 \left.\frac{\partial c_2}{\partial x}\right|_{x=x_*} = \frac{D_2}{x_0} c_0^{(2)} + \frac{2}{x_0} D_2 \sum_{m=1}^{\infty} \cos\left(x_m x_*\right) \left\{ c_0^{(2)} e^{-\lambda t} e^{-D_2 x_m^2 t} + \right.$$
$$\left. + \frac{1}{l x_m} \int_0^t \tilde{g}_m(t') e^{-\left[D_2 x_m^2 + \lambda\right](t-t')} \left(1+2\sum_{j=1}^{\infty}(-1)^j \cos\left(y_j\left(y-L\right)\right)\ e^{-D_2 y_j^2 (t-t')}\right) dt' \right\};$$

in particular, the mass flow $J_{0x}^{(2)}(y,t)$ through the lower surface $x = x_0$

$$J_{0x}^{(2)}(y,t) = \frac{D_2}{x_0} c_0^{(2)} + \frac{2}{x_0} D_2 \sum_{m=1}^{\infty} (-1)^{m+1} \left\{ c_0^{(2)} e^{-\lambda t} e^{-D_2 x_m^2 t} + \right.$$
$$\left. + \frac{1}{l x_m} \int_0^t \tilde{g}_m(t') e^{-\left[D_2 x_m^2 + \lambda\right](t-t')} \left(1+2\sum_{j=1}^{\infty}(-1)^j \cos\left(y_j\left(y-L\right)\right) e^{-D_2 y_j^2 (t-t')}\right) dt' \right\};$$

the mass flow $J_{*y}^{(2)}(x,t)$ through the surface $y = y_*$

$$J_{*y}^{(2)}(x,t) = -D_2 \left.\frac{\partial c_2}{\partial y}\right|_{y=y_*} = \frac{4 D_2}{l x_0} \sum_{m=1}^{\infty} \sin(x_m x) \times$$
$$\times \int_0^t \tilde{g}_m(t') e^{-[D_2 x_m^2 + \lambda](t-t')} \sum_{j=1}^{\infty} (-1)^{j+1} \sin\left(y_j\left(y_* - L\right)\right)\ e^{-D_2 y_j^2 (t-t')} dt'. \qquad (22)$$

Thus, we have obtained the mass flow expressions through any surface of the body.

5. Numerical analysis of decaying mass flows through a cross-section of the body

In this section, we explore decaying particle flow graphic distributions through the vertical cross-sections of the body: $J^{(1)}_{*\zeta}(\xi,\tau)$ in the middle of the domain Ω_1 at ζ_* =0.05 and $J^{(2)}_{*\zeta}(\xi,\tau)$ in the middle of the domain Ω_2 at ζ_* =0.55. Numerical calculations are performed according to the formulae (21) and (22) respectively, in dimensionless variables [8]:

$$\xi = (k/D_1)^{1/2}\, x, \quad \zeta = (k/D_1)^{1/2}\, y, \quad \tau = kt, \tag{23}$$

where k is the coefficient of the dimension $[c^{-1}]$ [5]. Calculations are accurate to $\varepsilon = 10^{-7}$. Herewith, the following settings are taken as basic: $\widehat{\lambda} = k/\lambda = 0.1$, $\xi_0 = (k/D_1)^{1/2} x_0 = 10$, $\Lambda = (k/D_1)^{1/2} L = 0.1$, $\gamma = (k/D_1)^{1/2} l = 0.9$, $d = D_2/D_1 = 0.01$, $\eta_1/\eta_2 = 0.1$, $c_0^{(2)}/c_0^{(1)} = 0.1$, $\widehat{v} = 0.2$, $\tau = 5$. Figs. 2–7 illustrate the distribution of the functions $J^{(i)}_{*\zeta}(\xi,\tau)$ depending on different values of the problem parameters. Fig. 2 illustrates the $J^{(i)}_{*\zeta}(\xi,\tau)$ distributions at different instants of dimensionless time τ. In Fig. 2, *a*, the curves 1–3 correspond to the values $\tau = 1; 5; 50$, and in Fig. 2, *b*, the curves 1–5 correspond to $\tau = 1; 5; 10; 30; 50$ for the convective velocity $\widehat{v} = 0.2$.

Figs. 3–4 show the $J^{(i)}_{*\zeta}(\xi,\tau)$ distributions depending on the coefficient of convective velocity in the domain Ω_1. Fig. 3 demonstrates the distribution of decaying admixture flows in thin channels, and Fig. 4 shows it in the domain of basic material. Figs. *a* are given for the small values of convective velocity, and Figs. *b* are for the large ones. Here, the curves 1–5 correspond to $\widehat{v} = 0.01; 0.02; 0.03; 0.04; 0.05$ in Fig. *a* and $\widehat{v} = 0.1; 0.2; 0.3; 0.4; 0.5$ in Fig. *b*.

Fig. 5 illustrates the behavior of the function $J^{(i)}_{*\zeta}(\xi,\tau)$ depending on the value of the coefficient of decay intensity of the migrating substance $\widehat{\lambda}$. Here, the curves 1–5 correspond to $\widehat{\lambda} = 0.01, 0.1, 0.2, 0.5, 1$. Figs. *a* are given for decaying impurity flow distributions in the domain Ω_1, and Fig. *b* are for the domain Ω_2.

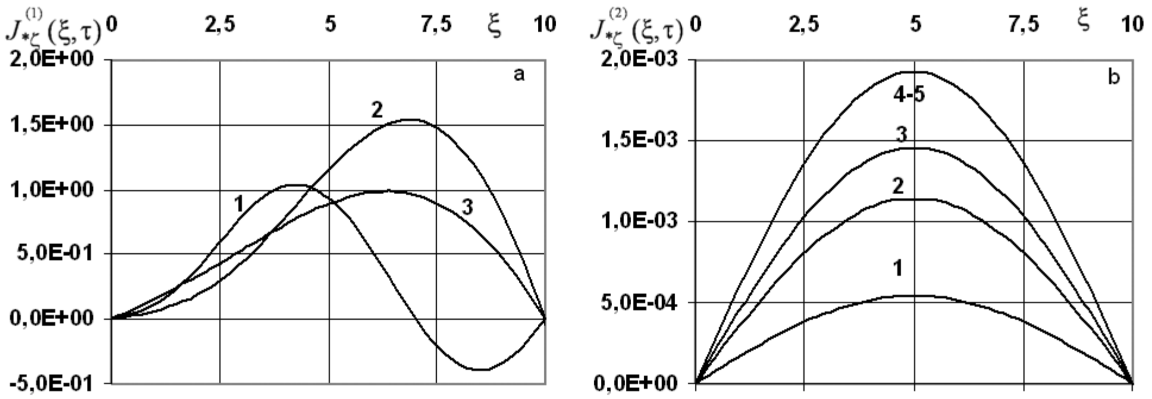

Fig. 2. Decaying particle flows $J^{(1)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_1 (*a*) and flows $J^{(2)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_2 (*b*) at different instants of dimensionless time τ for large values of convective velocity $\widehat{v}$.

Fig. 6 shows the effect of the ratio of the coefficients of the concentration dependence of the chemical potentials η_1/η_2, that determines the magnitude of the jump of the admixture concentration at the interface. Here, the curves 1–5 correspond to the values $\eta_1/\eta_2 = 0.17; 0.2; 0.25; 0.27; 0.3$. Fig. 7 shows the behavior of the function $J^{(i)}_{*\zeta}(\xi,\tau)$ depending on different values of the dimensionless diffusion coefficient $d = D_2/D_1$. Here, the curves 1–5 correspond to the values $d = 0.5, 0.6, 0.7, 0.8, 0.9$ in Fig. 7, *a* and $d = 0.1; 0.2; 0.3; 0.4; 0.5$ in Fig. 7, *b*.

Note that in the middle of the domain of thin channels Ω_1, the value of the admixture flow changes its sign, which means that the resulting flow changes its direction, increasing by times in the lower part

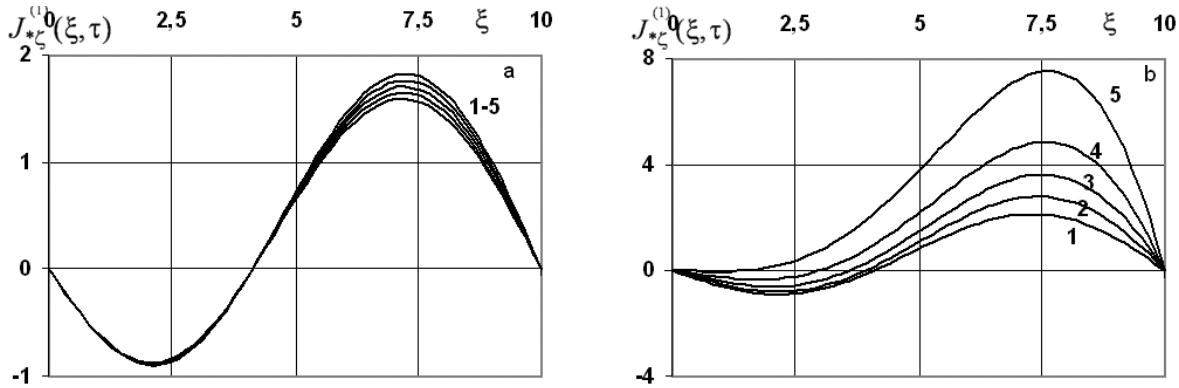

Fig. 3. Charts of decaying particle flows $J^{(1)}_{*\zeta}(\xi,\tau)$ in the middle of the thin channel depending on small (a) and large (b) values of the convective velocity $\widehat{v}$.

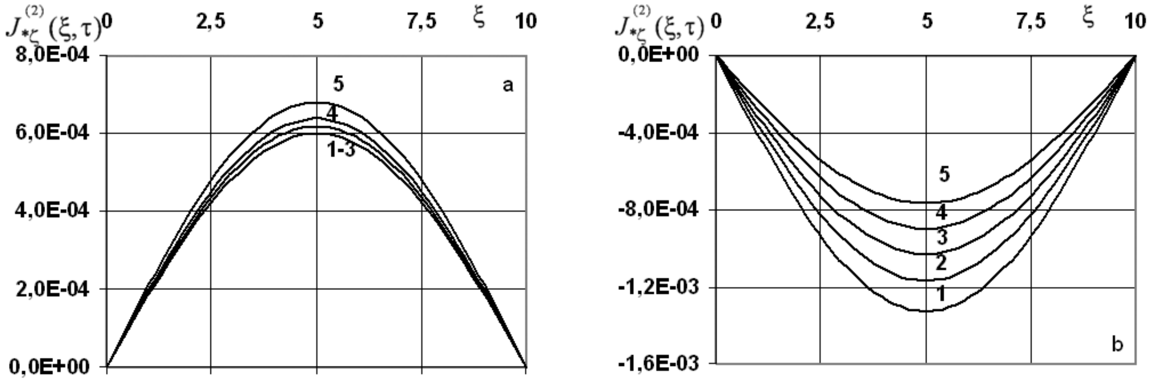

Fig. 4. Decaying particle flow $J^{(1)}_{*\zeta}(\xi,\tau)$ distribution in the middle of Ω_2 depending on small (a) and large (b) values of the convective velocity $\widehat{v}$.

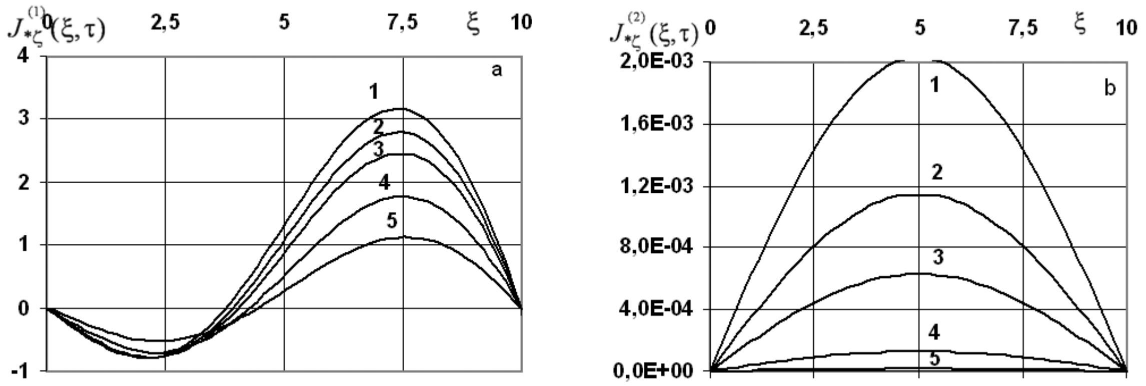

Fig. 5. Flow $J^{(1)}_{*\zeta}(\xi,\tau)$ distribution of the decaying admixture in the domain Ω_1 (a) and the flow $J^{(2)}_{*\zeta}(\xi,\tau)$ distribution in the domain Ω_2 (b) for different values of the coefficient of decay intensity of the migrating substance $\widehat{\lambda}$.

of the body (Figs. 5, a, 3, a). In the domain Ω_2, where the decaying particles are transferred by diffusion mechanism only, the flow distributions are symmetric relative to the layer middle (Figs. 2, b–7, b), also increasing in time until they put into the steady state (curves 4–5, Fig. 5).

The consideration of the convective component in thin channels leads to a substantial redistribution of decaying mass over the most part of the migration interval. With the increase of values of the

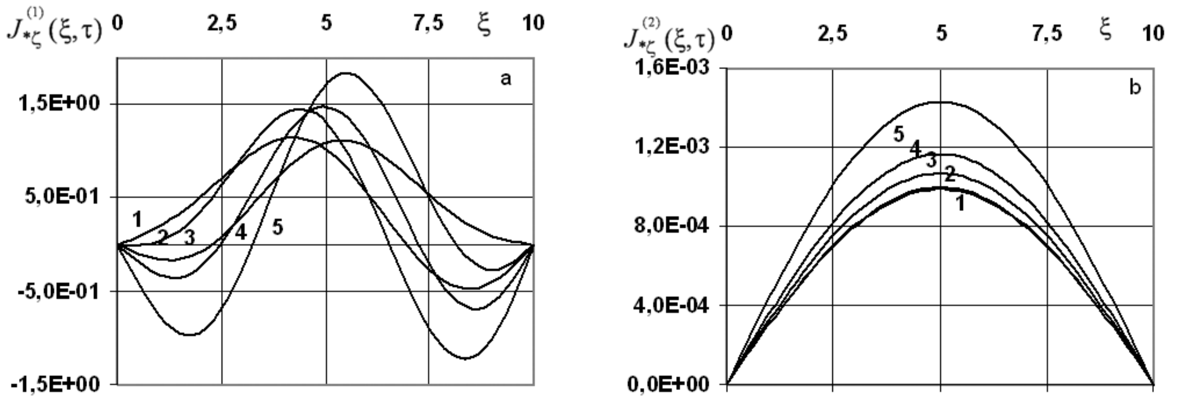

Fig. 6. Decaying particles flows $J^{(1)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_1 (a) and flows $J^{(2)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_2 (b) depending on the ratio of the coefficients of the concentration dependence of the chemical potentials η_1/η_2.

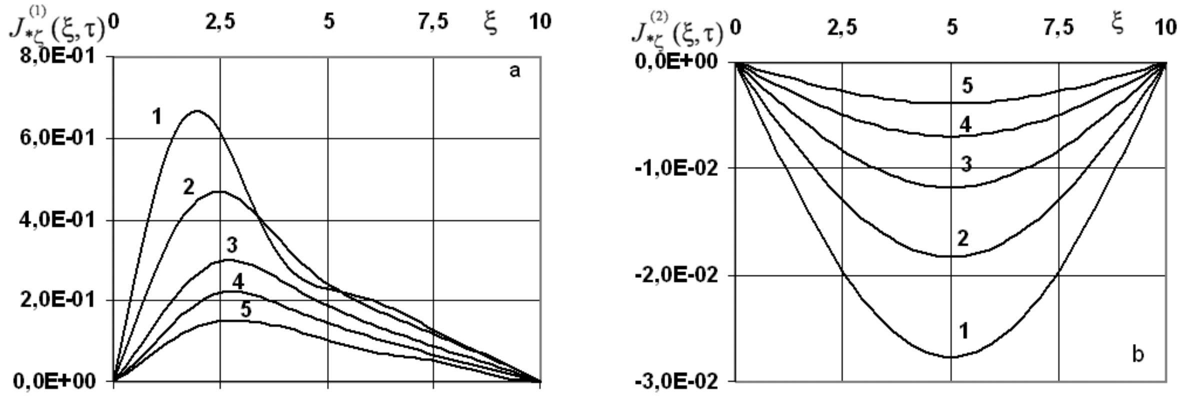

Fig. 7. Decaying particle flows $J^{(1)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_1 (a) and flows $J^{(2)}_{*\zeta}(\xi,\tau)$ in the middle of Ω_2 (b) depending on the diffusion coefficient d.

convective velocity, there is observed a significant decrease in the admixture flow through the middle of the thin channel. Thus, for large values of the convective velocity, the 5-times increase of the velocity leads to the 4-times decrease of the flow (curves 1, 5 in Fig. 4). In the domaine of basic material, the twice increase of convective velocity in thin channels causes the flow intensity decrease approximately by half (curves 2, 5 in Fig. 4, b).

The parameter of the decay intensity for the migrating substance $\widehat{\lambda}$ significantly influences the magnitude of decaying admixture flows $J^{(1)}_{*\zeta}(\xi,\tau)$ through vertical cross-sections of the domain Ω_1 and $J^{(2)}_{*\zeta}(\xi,\tau)$ of the domain Ω_2. Herewith, the greater the value of the coefficient of the decay intensity of the migrating substance $\widehat{\lambda}$ is, the smaller the resulting flow $J^{(i)}_{*\zeta}(\xi,\tau)$ value is (Fig. 5).

Other coefficients of the problem also significantly affect the behavior of the function $J^{(i)}_{*\zeta}(\xi,\tau)$. Such parameters of the problem as the ratio of diffusion coefficients $d = D_2/D_1$ and the ratio of the coefficients of mass source powers $c^{(2)}_0)/c^{(1)}_0$ on the surface of domains Ω_1 and Ω_2 significantly affect the magnitude of the decaying admixture flow $J^{(i)}_{*\zeta}(\xi,\tau)$. Herewith, with the increase of d, the value $|J_{*\zeta}(\xi,\tau)|$ decreases (Fig. 7). The admixture flow in thin channel reaches its peaks at small values of d right from the surface $\xi = 0$ where the source of decaying admixture mass is acting. Regarding the dependance of $J^{(i)}_{*\zeta}(\xi,\tau)$ on the mass source power on the surface $\xi = 0$, then the more value of the admixture concentration $c^{(2)}_0$ in the domain Ω_2 is, then the less the value of the function $|J_{*\zeta}(\xi,\tau)|$ in the domain Ω_1 is. In the domain Ω_2, the opposite pattern is observed: with the increase of $c_0^{(2)}/c_0^{(1)}$, the value of $|J_{*\zeta}^{(2)}(\xi,\tau)|$ also increases.

Note that among all the parameters of the problem, the ratio of the coefficients of the concentration dependence of the chemical potentials η_1/η_2, which determines the jump of the concentration function at the interface, has the most effect on the function of decaying admixture flow in thin channels. A small change in the value of this parameter leads to significant changes in the flow $J_{*\zeta}^{(1)}(\xi,\tau)$ distribution not only quantitatively but also qualitatively. Thus, with increasing values of η_1/η_2, the resulting flow changes its direction right from the surface, where there is a source of the admixture mass supply (curves 3–5, Fig. 6, *a*). With further increase in the values of η_1/η_2, we observe the appearance of the second local maximum of the function $J_{*\zeta}^{(1)}(\xi,\tau)$ approximately in the middle of the layer, and the appearance of the third local maximum near the lower boundary surface $\xi=\xi_*$ of the layer (curves 1–5, Fig. 6, *a*). In the domain Ω_2 this parameter affects the flow only quantitatively – for larger values of η_1/η_2 larger absolute values of $J_{*\zeta}^{(2)}(\xi)$ correspond (Fig. 6, *b*).

6. Modelling averaged decaying admixture concentration in the layer of regularly located thin channels

We can introduce the function of the total concentration of decaying substance averaged with respect to the variable y as follows:

$$\langle c(x,y,t)\rangle=\frac{1}{L+l}\int\limits_0^L c_1(x,y,t)dy+\frac{1}{L+l}\int\limits_L^{L+l} c_2(x,y,t)dy. \tag{24}$$

By substitution of the expressions (20), (21) for the concentration $c_1(x,y,t)$ in the domain Ω_1 and $c_2(x,y,t)$ in Ω_2 into the correlation (24), we obtain

$$\langle c(x,y,t)\rangle=\frac{1}{L+l}\left\langle Le^{-\lambda t}e^{v_Dx}c_0^{(1)}\frac{\operatorname{sh}v_D\left(x_0-x\right)}{\operatorname{sh}\left(v_Dx_0\right)}+lc_0^{(2)}\left(1-\frac{x}{x_0}\right)+\frac{2L}{x_0}e^{-\lambda t}e^{v_Dx}\sum_{n=1}^{\infty}\sin(x_nx)\times\right.$$
$$\times\left[-\frac{c_0^{(1)}x_n}{v_D^2+x_n^2}e^{-D_1\left(v_D^2+x_n^2\right)t}+\frac{1}{L}\int\limits_0^t\tilde g_n(t')e^{\lambda t'}e^{-D_1\left(v_D^2+x_n^2\right)(t-t')}dt'\right]-$$
$$\left.-\frac{2l}{x_0}\sum_{m=1}^{\infty}\sin(x_mx)\left[\frac{1}{x_m}c_0^{(2)}e^{-\lambda t}e^{-D_2x_m^2t}+\frac{1}{l}\int\limits_0^t\tilde g_m(t')e^{-\left[D_2x_m^2+\lambda\right](t-t')}dt'\right]\right\rangle. \tag{25}$$

If we introduce the parameter $\alpha=l/L$, we will rewrite (25) in the form

$$\langle c(x,y,t)\rangle=\frac{1}{1+\alpha}e^{-\lambda t}e^{v_Dx}c_0^{(1)}\frac{\operatorname{sh}(v_D(x_0-x))}{\operatorname{sh}\left(v_Dx_0\right)}+\frac{2c_0^{(2)}}{(1+\alpha)}\left(1-\frac{x}{x_0}\right)-\frac{2e^{-\lambda t}e^{v_Dx}}{x_0(1+\alpha)}\sum_{n=1}^{\infty}\sin(x_nx)\times$$
$$\times\left[\frac{c_0^{(1)}x_n}{v_D^2+x_n^2}e^{-D_1\left(v_D^2+x_n^2\right)t}-\frac{\alpha}{l}\int\limits_0^t\tilde g_n^{\alpha}(t')e^{\lambda t'}e^{-D_1\left(v_D^2+x_n^2\right)(t-t')}dt'\right]-\frac{2}{x_0}\frac{\alpha}{1+\alpha}\sum_{m}^{\infty}\sin(x_mx)\times$$
$$\times\left[\frac{c_0^{(2)}}{x_m}e^{-\lambda t}e^{-D_2x_m^2t}+\frac{1}{l}\int\limits_0^t\tilde g_m^{\alpha}(t')e^{-\left[D_2x_m^2+\lambda\right](t-t')}dt'\right],$$

where

$$\tilde{g}_n^\alpha(t') = \frac{-\frac{\eta_1}{\eta_2} D_1 c_0^{(1)} x_n + \frac{2}{x_0} D_2 c_0^{(2)} \sum_{m=1}^{\infty} x_m B_{n,m} E_{n,m}\left(t-t'\right)}{\frac{\eta_1}{\eta_2}\frac{\alpha}{l}\Theta_0\left(0, e^{-D_1\frac{k^2\pi^2\alpha^2}{l^2}(t-t')}\right) + \frac{4}{x_0^2 l}\sum_{m=1}^{\infty}\Phi_m\left(t-t'\right) A_{n,m} B_{n,m} e^{D_1\left(v_D^2+x_n^2\right)(t-t')}},$$

$$\tilde{g}_m^\alpha(t') = \frac{2}{x_0}\sum_{n=1}^{\infty} A_{n,m}\tilde{g}_n^\alpha(t'),$$

here $R_n^\alpha = \frac{1}{\psi_n}\operatorname{cth}\left(\psi_n \frac{l}{\alpha}\right) + \frac{1}{\psi_n^2}\left(1-\frac{\alpha}{l}\right)$.

Let l tend to 0 with $\alpha \equiv$ const. Then we have to determine

$$\lim_{l\to 0}\langle c(x,y,t)\rangle = \frac{1}{1+\alpha} c_0^{(1)} e^{-\lambda t} e^{v_D x}\frac{\operatorname{sh}(v_D(x_0-x))}{\operatorname{sh} v_D x_0} + \frac{\alpha}{1+\alpha} c_0^{(2)}\left(1-\frac{x}{x_0}\right) -$$
$$- \frac{2e^{-\lambda t}e^{v_D x}}{x_0(1+\alpha)}\sum_{n=1}^{\infty}\sin(x_n x)\left\{\frac{c_0^{(1)} x_n}{v_D^2+x_n^2} e^{-D_1\left(v_D^2+x_n^2\right)t} - \alpha\int_0^t e^{\lambda t'} e^{-D_1\left(v_D^2+x_n^2\right)(t-t')}\lim_{l\to 0}\frac{1}{l}\tilde{g}_n^\alpha(t')dt'\right\} -$$
$$- \frac{2\alpha}{(1+\alpha)x_0}\sum_{m=1}^{\infty}\sin(x_m x)\left[\frac{c_0^{(2)}}{x_m} e^{-\lambda t} e^{-D_2 x_m^2 t} + \int_0^t e^{-\left(D_2 x_m^2+\lambda\right)(t-t')}\lim_{l\to 0}\frac{1}{l}\tilde{g}_m^\alpha(t')dt'\right]. \quad (26)$$

Since in the equation (26) $g_m^0(t') = \lim_{l\to 0}\frac{1}{l}\tilde{g}_m^\alpha = \frac{2}{x_0}\lim_{l\to 0}\sum_{m=1}^{\infty}\frac{A_{n,m}}{l}\tilde{g}_n^\alpha(t)$, then we seek $\lim_{l\to 0}\frac{1}{l}\tilde{g}_n^\alpha$ only. Thus, we have

$$\lim_{l\to 0}\frac{1}{l}\tilde{g}_n^\alpha = \lim_{l\to 0}\frac{-\frac{\eta_1}{\eta_2} D_1 c_0^{(1)} x_n + \frac{2}{x_0} D_2 c_0^{(2)}\sum_{m=1}^{\infty} x_m B_{n,m} E_{n,m}\left(t-t'\right)}{\alpha\frac{\eta_1}{\eta_2}\left(1+2\sum_{k=1}^{\infty} e^{-D_1\frac{k^2\pi^2\alpha^2}{l^2}(t-t')}\right) + \frac{4}{x_0^2} S_{n,m}},$$

where $S_{n,m} = \sum_{m=1}^{\infty}\Phi_m\left(t-t'\right) A_{n,m} B_{n,m} e^{D_1\left(v_D^2+x_n^2\right)(t-t')}$.

Having obtained the limit at $l \to 0$, we can obtain

$$g_n^0(t') \equiv \lim_{l\to 0}\frac{1}{l}\tilde{g}_n^\alpha = \frac{-\frac{\eta_1}{\eta_2} D_1 c_0^{(1)} x_n + \frac{2}{x_0} D_2 c_0^{(2)}\sum_{m=1}^{\infty} x_m B_{n,m} E_{n,m}\left(t-t'\right)}{\alpha\frac{\eta_1}{\eta_2} + \sum_{m=1}^{\infty} e^{-D_2 x_m^2(t't')} A_{n,m} B_{n,m} e^{D_1\left(v_D^2+x_n^2\right)(t-t')}}. \quad (27)$$

Using (27), we have respectively

$$\lim_{l\to 0}\langle c(x,y,t)\rangle = \frac{1}{1+\alpha} e^{-\lambda t} e^{v_D x} c_0^{(1)}\frac{\operatorname{sh} v_D(x_0-x)}{\operatorname{sh}(v_D x_0)} + \frac{\alpha c_0^{(2)}}{1+\alpha}\left(1-\frac{x}{x_0}\right) -$$
$$- \frac{2e^{-\lambda t}e^{v_D x}}{x_0(1+\alpha)}\sum_{n=1}^{\infty}\sin(x_n x)\left[\frac{c_0^{(1)} x_n}{v_D^2+x_n^2} e^{-D_1\left(v_D^2+x_n^2\right)t} - \alpha\int_0^t e^{\lambda t'} e^{-D_1\left(v_D^2+x_n^2\right)(t-t')} g_n^0(t')dt'\right] -$$

$$-\frac{2\alpha}{x_0(1+\alpha)}\sum_{m=1}^{\infty}\sin(x_m x)\left[\frac{c_0^{(2)}}{x_m}e^{-\lambda t}e^{-D_2 x_m^2 t}+\int_0^t e^{-\left[D_2 x_m^2+\lambda\right](t-t')}g_m^0(t')dt'\right]. \qquad (28)$$

Note that if it is the parameter $\bar{\alpha}=L/l$ that is introduced, and L is let tend to 0 with $\bar{\alpha}\equiv$const, then we will obtain the same formula (28).

For example, in Fig. 8 we give typical admixture concentration distributions of decaying substance, which is averaged over the width of the selected element of the body, and they are calculated according to the formula (25). Here are shown averaged concentration distributions calculated for different values of convective velocity $\widehat{v}=0.1, 0.2, 0.3; 0.4$ (curves 1a-4a) and $\widehat{v}$=0.1, 0.2, 0.3; 0.4 (curves 1b–5b) for the basic values of the problem parameters.

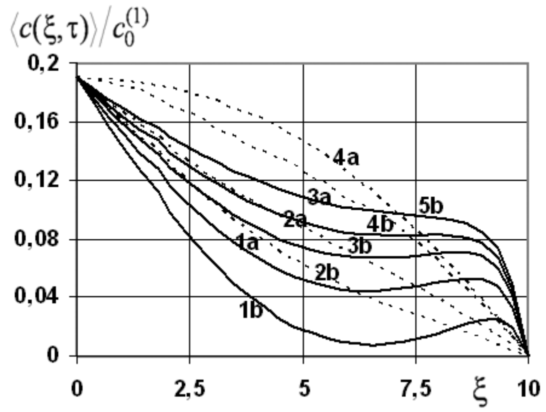

Fig. 8. Dependence of the function of averaged concentration on small (curves a) and large (curves b) values of the convective velocity $\widehat{v}$.

Note that the behaviour and values of the function of the total concentration of decaying substance averaged over the width of the selected element of the body are affected the most by the coefficient of convective velocity, herewith for small values of $\widehat{v}$, the function $\langle c(\xi,\tau)\rangle$ is monotonically decreasing, (curves 1a-3a, Fig. 8); with the increase of the magnitude of values $\widehat{v}$ the averaged concentration values increase over the whole interval, and the function $\langle c(\xi,\tau)\rangle$ becomes convex (curve 4a, Fig. 8). Further increase in the coefficient of convective velocity leads to a decrease in the decaying admixture concentration averaged over the width of the selected element of the body, but the greatest decline is observed in the middle of the layer, herewith a local maximum of $\langle c(\xi,\tau)\rangle$ appears near the lower surface of the body (curve 1b, Fig. 8).

7. Conclusions

In this work the flows of decaying admixture through arbitrary cross-sections of the body of a regular structure at periodic location of thin channels of fast movement of decaying admixture particles are investigated. The formulae of these flows are obtained on the basis of exact analytical solutions of the contact initial boundary value problem of convective diffusion of a decaying substance, in which there are taken into account the convective mechanism of substance transfer by thin channels. These solutions are obtained using the method of integral transformations with respect to the spatial variables separately in each of the contacting domains. The correlations between these integral transformations are obtained using contact conditions; the Volterra integral equation of the first kind is obtained to determine the mass flow at the interface. It is shown that the decay of the migrating substance does not affect the flow rate through the interface of the contacting domains which compose the body. However, it significantly affects the quantitative flow distribution inside of the contacting domains. This is particularly true to the domain of basic material, where the twice decrease of the magnitude of the coefficient of decay intensity of the migrating substance leads to the same decline of the admixture flow intensity.

Note also that the applied method of constructing exact solutions of the contact initial boundary value problem of the nonstationary convective diffusion of decaying substance does not use conditions on the size of the contacting domains, that means it can be applied to bodies with commensurate sizes of the contacting domains, as well as in the case when the width of the domain where mass transfer occurs both diffusive and convective mechanisms is much larger or smaller than the width of the domain in which the decaying admixture diffusion only is allowed for.

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Комп'ютерне моделювання потоків розпадної речовини в шарі з періодично розташованими тонкими каналами

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В роботі досліджено процеси конвективної дифузії домішкової речовини у двофазних структурах з періодично розташованими тонкими каналами з урахуванням натурального розпаду мігруючої речовини. За допомогою відповідних інтегральних перетворень окремо в контактуючих областях отримано розв'язок контактно-крайових задач конвективної дифузії розпадної речовини. Зв'язок між цими інтегральними перетвореннями знайдений з використанням неідеальних контактних умов, сформульованих на функцію концентрації. Знайдено та досліджено вирази для потоків розпадних частинок через довільні перерізи тіла та проведено їхній числовий аналіз в середині тонких каналів та основного матеріалу. Показано, що інтенсивність розпаду мігруючої речовини особливо впливає на розподіли потоків в області основного матеріалу.

Ключові слова: *дифузія, конвекція, розпад домішки, регулярна структура, тонкий канал, потік маси*

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