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ON THE GRAVITATIONAL POTENTIAL ENERGY OF THE EARTH

Для оцінювання гравітаційної потенціальної енергії Землі E використано 3D-розподіл густини еліпсоїдальної планети разом з його оцінкою точності. Саме використання останньої дало змогу виконати оцінювання E на основі лише радіального розподілу густини у вигляді її неперервних та кусково-неперервних моделей: Лежандра-Лапласа, Роша, Булларда і Гаусса. В результаті отримана нерівність для E з верхнею границею E_H для однорідного розподілу і нижнею границею E_{Gauss}, яка відповідає розподілу Гаусса для густини Землі. Головні оцінки E дають гарне погодження з E_{Gauss}: як у випадку E, яке базується на моделі Роша з 6 головними стрибками густини, так і оцінки E, що відповідають 4 найпростішим моделям з одним стрибком густини на границі ядро-мантія.

Для оценки гравитационной потенциальной энергии Земли E использовано 3D-распределение плотности внутри эллипсоидальной планеты совместно с ее оценкой точностью. Именно применение последней дало возможность выполнить оценивание E только на базе радиального распределения плотности в виде ее непрерывных и кусочно-непрерывных моделей: Лежандра-Лапласа, Роша, Булларда и Гаусса. В результате получено неравенство для E с верхним пределом E_H для однородной планеты и нижним пределом E_{Gauss}, соответствующим Гауссовому распределению. Главные оценки E дают прекрасное согласование с E_{Gauss}, включая значение E, основанное на модели Роша с 6 главными скачками плотности внутри Земли, и оценки E, отвечающие простейшим 4 моделям с одним скачком на границе ядро-мантия.

1 Introduction. Determination of the Earth's volume density distribution $\delta(\rho,\vartheta,\lambda)$ from external potential data requires a solution of the known inverse problem of the Newtonian potential. If the planet's gravitational potential energy *E* and density at the surface are accepted as additional information, this problem transforms from an improperly posed to a properly posed problem with its possible solution for the 3D density $\delta(\rho,\vartheta,\lambda)$ through the three-dimensional Cartesian moments (Mescheryakov, 1977). According to Gauss (1840) the search of the stationary value *E* can be treated as one of central subjects of the potential theory. A remarkable summary of the Gauss' problem reads: "minimum and maximum potential energy correspond to physically (for the Earth) meaningless cases: a surface distribution and a mass point. The 'true' Earth lies somewhere in between" (Moritz, 1990). It is obvious that the potential energy *E* can be estimated from the density and internal gravitational potential. However only few *E*-values for the homogeneous Earth (Mescheryakov, 1973; Rubincam, 1979; Moritz, 1990) and the planet differentiated into homogeneous mantle and homogeneous core (Rubincam, 1979) are found in the literature. Thus, the question remains: how can we evaluate better this 'true' Earth and the corresponding potential energy *E*.

This study focuses on (a) the determination of the Earth's global density distribution and (b) the estimation of the gravitational potential energy *E* using continuous and piecewise density models. The Earth's mass and principal moments of inertia represent initial information for the unique solution of the restricted Cartesian moments problem providing in this way the density $\delta(\rho,\vartheta,\lambda)$ and the potential energy *E*. The principal moments of inertia given in (Marchenko, 2007) were used for the computation of the 3D global density $\delta(\rho,\vartheta,\lambda)$.

It should be pointed out, that accuracy of the global density and potential energy was derived especially to restrict the possible solution domain in such a way that a reasonable solution may be selected either from 3D-spatial or radial density inside the ellipsoidal or spherical planet.

2. The Earth's global density distribution. Let us consider the mathematical model of the 3D global density distribution $\delta(\rho,\vartheta,\lambda)$ derived by (Mescheryakov et al., 1977) inside the Earth having a shape of the ellipsoid of revolution with the flattening *f* and the semimajor axis *a*. According to Mescheryakov (1991) the exact but restricted by the order 2 solution of the three-dimension Cartesian moments problem for $\delta(\rho,\vartheta,\lambda)$ reads

$$\delta(\rho,\vartheta,\lambda) = \delta(\rho)_{\mathrm{R}} + \Delta\delta(\rho,\vartheta,\lambda)\,, \tag{1}$$

$$\Delta\delta(\rho,\vartheta,\lambda) = \Delta K + \rho^2(\Delta K_1 \sin^2\vartheta\cos^2\lambda + \Delta K_2 \sin^2\vartheta\sin^2 + \Delta K_3\cos^2\vartheta)\,, \tag{2}$$

where $\delta(\rho)_{\mathrm{R}}$ is the piecewise reference radial density model with radial density jumps such as PREM (Dziewonski and Anderson, 1981), $\Delta\delta(\rho,\vartheta,\lambda)$ is some anomalous density with the following components

$$\left.\begin{aligned} \Delta K &= \frac{5}{4}\delta_m[5\Delta I_{000} - 7(\Delta I_{200} + \Delta I_{020} + \frac{\Delta I_{002}}{\chi^2})],\\ \Delta K_1 &= \frac{35}{4}\delta_m(3\Delta I_{200} + \Delta I_{020} + \frac{\Delta I_{002}}{\chi^2} - \Delta I_{000}),\\ \Delta K_2 &= \frac{35}{4}\delta_m(\Delta I_{200} + 3\Delta I_{020} + \frac{\Delta I_{002}}{\chi^2} - \Delta I_{000}),\\ \Delta K_3 &= \frac{35}{4}\delta_m(\Delta I_{200} + \Delta I_{020} + 3\frac{\Delta I_{002}}{\chi^2} - \Delta I_{000}), \end{aligned}\right\} \qquad \left.\begin{aligned} \Delta I_{000} &= I_{000} - I_{000}^R, \quad \Delta I_{200} = I_{200} - I_{200}^R,\\ \Delta I_{020} &= I_{020} - I_{020}^R, \quad \Delta I_{002} = I_{002} - I_{002}^R. \end{aligned}\right\} \tag{3}$$

In the relationships above χ=1–*f*, the dimensionless Cartesian moments I_{000}, I_{200}, I_{020}, and I_{002} ($I_{100} = I_{010} = I_{001} = 0$) of the density of a gravitating body (Grafarend et al., 2000) can be computed via the Earth's mass *M* and dimensionless principal moments of inertia *A*, *B*, and *C* normalized by $1/Ma^2$:

$$\left.\begin{aligned} I_{000} &= 1,\\ I_{200} &= (B + C - A)/2,\\ I_{020} &= (A - B + C)/2,\\ I_{002} &= (A + B - C)/2, \end{aligned}\right\} \qquad \left.\begin{aligned} A &= \sqrt{5}\bar{A}_{20}(1 - 1/H_D) - \sqrt{15}\bar{A}_{22}/3,\\ B &= \sqrt{5}\bar{A}_{20}(1 - 1/H_D) + \sqrt{15}\bar{A}_{22}/3,\\ C &= -\sqrt{5}\bar{A}_{20}/H_D. \end{aligned}\right\} \tag{4}$$

The reference model $\delta(\rho)_{\mathrm{R}}$ includes individual information about density jumps, the mean density δ_m^R, and the mean moment of inertia I_m^R, which have been selected preliminary for the construction of the radial profile $\delta(\rho)_{\mathrm{R}}$. In contrast to Mescheryakov (1991) [Eqs. (1 – 3)] the Cartesian moments I_{000}^R, I_{200}^R, I_{020}^R, and I_{002}^R of the reference density $\delta(\rho)_{\mathrm{R}}$ were derived here for one common set of the conventional constants δ_m and I_m of the model (1) and density jumps entering into $\delta(\rho)_{\mathrm{R}}$:

$$\left.\begin{aligned} I_{000}^R &= \frac{\delta_m^R}{\delta_m},\\ I_{200}^R &= I_{020}^R = \frac{3I_m^R\delta_m^R}{2\delta_m(\chi^2+2)},\\ I_{002}^R &= \frac{3\cdot\chi^2 I_m^R\delta_m^R}{2\cdot\delta_m(\chi^2+2)}, \end{aligned}\right\} \qquad \left.\begin{aligned} \delta_m^R &= 3\int_0^1 \delta(\rho)_{\mathrm{R}}\,\rho^2 d\rho,\\ I_m^R &= \frac{2(\chi^2+2)}{3\delta_m^R}\int_0^1 \delta(\rho)_{\mathrm{R}}\,\rho^4 d\rho. \end{aligned}\right\} \tag{5}$$

Thus, in these formulae ρ ($0 \le \rho \le 1$) is the relative distance from the origin of a coordinate system to an internal current point; ϑ and λ are the polar distance and longitude of this point; δ_m is the convenient mean density; H_D is the dynamical ellipticity; $\bar{A}_{20}, \bar{A}_{22}$ are the fully normalized (non-zero) harmonic coefficients adopted here as Stokes constants in the principal axes system $O\bar{A}\bar{B}\bar{C}$. Therefore, this 3D global density [Eq.(1)] is given in the geocentric coordinate system of the principal axes of inertia and agreed with the Earth's mass and the principal moments of inertia to preserve in this way the external gravitational potential from zero to second degree/order, H_D, the flattening f, and density jumps.

The radial density $\delta(\rho)_{\mathrm{R}}$ is also treated within the ellipsoid of revolution if we use the formula $r_e = R(1-2f\cdot P_2(\cos\vartheta)/3)$ for the radius vector r_e by neglecting f^2 (Moritz, 1990), where $P_2(\cos\vartheta)$ is the 2nd-degree Legendre polynomial. This formula results from the average of r_e over the unite sphere that gives the mean radius R=6371 km. According to Mescheryakov (1991) Eqs. (1 – 2) are valid for a homothetic stratification when f=const inside the ellipsoidal Earth. Hence, if the set of the internal ellipsoidal surfaces $\tilde{r}_e$ is labeled by the associated mean radius r of a sphere we have

$$\tilde{r}_e = r\left[1-\frac{2}{3}f\cdot P_2(\cos\vartheta)\right] \Rightarrow \rho = \frac{r}{R} = \frac{\tilde{r}_e}{r_e}. \tag{6}$$

By averaging $\delta(\rho,\vartheta,\lambda)$ over ellipsoidal surfaces we define the piecewise radial density $\delta(\rho)$ as

$$\left.\begin{aligned} \delta(\rho) &= \delta(\rho)_{\mathrm{R}} + \left[\Delta K + \rho^2\Delta D\right], \\ \Delta D &= \frac{35}{12}\delta_m\left[5\left(\Delta I_{200} + \Delta I_{020} + \frac{\Delta I_{002}}{\chi^2}\right) - 3\Delta I_{000}\right], \end{aligned}\right\} \tag{7}$$

with the treatment of the reference density $\delta(\rho)_{\mathrm{R}}$ within the ellipsoidal Earth. Since the radius ρ is constant for each $\tilde{r}_e$, the densities $\delta(\rho)_{\mathrm{R}}$ and $\delta(\rho)$ are also constant by Eqs. (7) at the surface (6).

Accuracy estimation of the 3D continuous global density was derived from error propagation, keeping in mind that information about accuracy of the mean density δ_m^R, the mean moment of inertia I_m^R, and density jumps in different piecewise radial models $\delta(\rho)_{\mathrm{R}}$ (such as PREM) are not found in literature or were not easily accessible to the author. For this reason we will consider the reference density $\delta(\rho)_{\mathrm{R}}$ as some exact constituent or "normal density". Hence, the variance-covariance matrix of the principal moments of inertia, accuracy of the mean density σ_{δ_m}, and accuracy of the flattening σ_f were chosen as initial information (Table 1).

The accuracy σ_{δ_m} of the mean density δ_m requires additional remarks because this value represents a scale factor of the considered theory. If $\delta_m = 5.514\,\mathrm{g/cm^3}$ and the gravitational constant $G = (6.673 \pm 0.010)\cdot 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$ suggested by the IERS Conventions 2003 (McCarthy and Petit, 2004) are selected, we get $\sigma_{\delta_m} = 0.08\,\mathrm{g/cm^3}$. According to the IAG recommendations for G and GM (Table 1) another mean density $\delta_m = (5.5145 \pm 0.0026)\,\mathrm{g/cm^3}$ finally was adopted.

Table 1

Initial parameters and their accuracy

Reference	Adopted parameters
Groten, 2004	G=(6.67259±0.0003)·10^{-11} $\mathrm{m^3kg^{-1}s^{-2}}$
Groten, 2004	GM=(398600.4415±0.0008)·$10^9\mathrm{m^3s^{-2}}$
Marchenko, 2007	A=0.3296127±0.0000005
Marchenko, 2007	B=0.3296200±0.0000005
Marchenko, 2007	C=0.3306990±0.0000005
Marchenko and Schwintzer, 2003	$1/f$=298.25650 ± 0.00001

Thus, the global density distribution and accuracy at different depths were based on the value $\delta_m = (5.5145 \pm 0.0026)\,\mathrm{g/cm^3}$, the flattening f, and the principal moments of inertia A, B, and C from Table 1. The principal moments of inertia (given here in the zero frequency tide system) are results from the adjustment of the 2nd-degree harmonic coefficients of 6 gravity field models and 7 values H_D of the dynamical ellipticity all transformed to the common value of precession constant at epoch J2000. The reference radial density profile $\delta(\rho)_{\mathrm{R}}$ in Eq. (1) was selected in the form of the simple piecewise Roche's law separated into seven basic shells (Marchenko, 2000), which is slightly different from PREM.

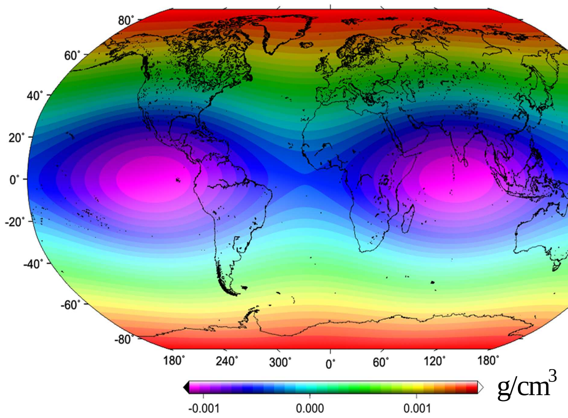

Fig. 1. Density anomalies [g/cm³] $\Delta\delta(\rho,\vartheta,\lambda)$ [Eq.(2)] at the mantle/crust boundary (r=6346.6 km)

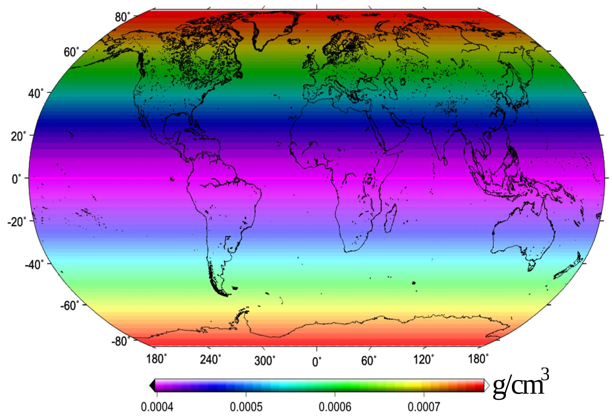

Fig. 2. Accuracy $\sigma_{\delta(\rho,\vartheta,\lambda)}$ [g/cm³] of the continuous 3D density distribution at the mantle/crust boundary

Therefore, with $\delta(\rho)_{\mathrm{R}}$ known as exact constituent, the accuracy estimation $\sigma_{\delta(\rho,\vartheta,\lambda)}$ of the 3D continuous global density distribution $\tilde{\delta}(\rho,\vartheta,\lambda)$ (based only on the Earth's mechanical parameters) and lateral density heterogeneities [Eq. (2)] are straightforward. Comparison of these lateral density anomalies $\Delta\delta(\rho,\vartheta,\lambda)$ (Fig. 1) with the accuracy $\sigma_{\delta(\rho,\vartheta,\lambda)}$ of the continuous constituent at the same depths (Fig. 2) leads generally at least to values of the same order in uncertainties and density heterogeneities taken for various depths. Because discussed uncertainties are increasing when radius ρ is decreasing to zero we will use below only radial density models for further determination of the Earth's gravitational potential energy.

3. Estimation of the gravitational potential energy. As well-known the computation of the gravitational potential energy is based on the following expression (Moritz, 1990):

$$E = -\frac{1}{2}\int_{\tau} \delta \cdot V_i \cdot d\tau\,, \qquad (8)$$

where δ is the Earth's density, V_i is the internal gravitational potential, and τ is the planet's volume.

Table 2

Expressions for different radial density models

Model	Mathematical expression
Homogeneous planet	$\delta(\rho) = \delta_m = \mathrm{const}$
Legendre-Laplace law	$\delta(\rho) = \delta_0 \sin(\gamma\rho)/(\gamma\rho)$
Roche's law	$\delta(\rho) = a + b\rho^2$
Bullard's model	$\delta(\rho) = a + b\rho^2 + c\rho^4$
Gauss' model (Marchenko, 2000)	$\delta(\rho) = \delta_0 \exp(-\beta^2\rho^2)$

For the determination of the potential energy E we will examine additionally to the homogeneous Earth the following radial-only continuous density profiles: Legendre-Laplace law, Roche's law (as solutions of the Clairaut's equation), Bullard's model, and Gaussian (normal) distribution (see Fig. 3). Therefore, in order to determine the gravitational potential energy E we use density laws from Table 2 initially for the spherical Earth. The parameters of the simplest density models (Fig. 3) listed in Table 2 were derived in the closed form (Marchenko, 2000) from the solution of the inverse problem based on the well-known conditions to keep δ_m, I_m, and the density δ_s at the Earth' surface (Moritz, 1990). The parameters δ_0 and a represent the density at the origin (Table 2 and Table 3). Initially we derive relationships for the internal potential V_i corresponding to these density laws. Then, applying Eq. (8) to the density models from Table 2 and internal potentials we find final expressions given in Table 3 for the estimation of the potential energy E of the spherical Earth.

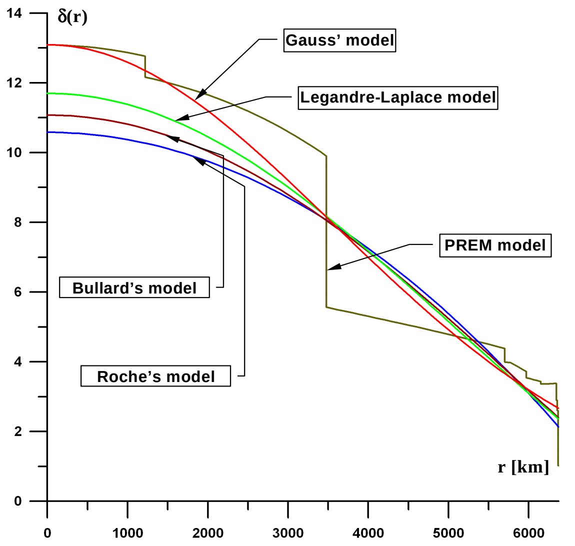

Fig. 3. Legendre-Laplace, Roche, Bullard, and Gauss continuous densities compared with PREM [g/cm³]

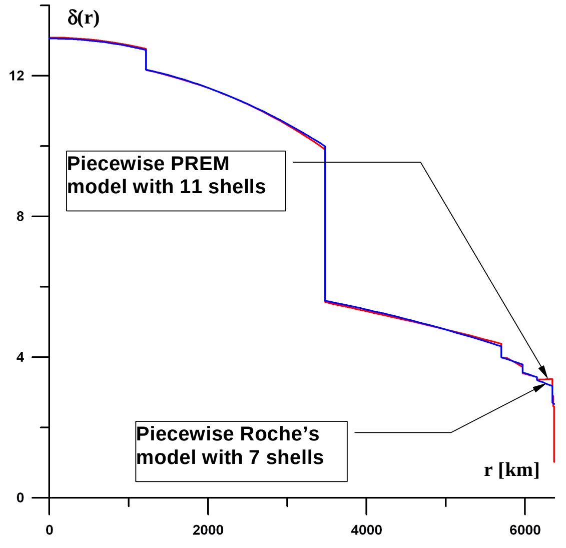

Fig. 4. Piecewise Roche-density model with 7 shells compared with PREM-density model $\delta(\rho)$ [g/cm³]

With adopted δ_m and the dimensionless mean moment of inertia $I_m = 0.3299773 \pm 0.0000005$ (Table 1) numerically we get estimations of the energy E given in Table 4, which includes E-estimates given by Mescheryakov (1973) and Rubincam (1979) for further comparisons. Thus, there are two limits for all computed E:

$$E_{\text{Gauss}} \le E_{\text{Earth}} \le E_{\text{Homogeneous}}\,. \qquad (9)$$

The upper limit E_{H} agrees with the homogeneous Earth. The minimum amount E_{Gauss} corresponds to the Gauss' model. Such assertion has an evident mathematical explanation. The first term of the Taylor series expansion of E_{Gauss} from Table 3 represents the gravitational potential energy of the homogeneous Earth E_{H}. Globally speaking every model from Table 3 includes the main term equal to E_{H}. But the sum of other terms with E_{H} gives a smaller E than the value E_{H}.

Table 3

The gravitational potential energy *E* of the spherical Earth for different radial density models

Model	Mathematical expression
Homogeneous planet	$E = -16\pi^2 G\delta_m^2 R^5/15$
Legendre-Laplace law	$E = -\dfrac{4\pi^2 G\delta_0^2 R^5}{\gamma^4}\left(2\cos^2\gamma - \dfrac{3\sin\gamma\cos\gamma}{\gamma} + 1\right)$
Roche's law	$E = -\dfrac{16\pi^2 GR^5}{315}(21a^2 + 24ab + 7b^2)$
Bullard's model	$E = -\dfrac{16\pi^2 GR^5}{135135}\left[286a(36b+25c) + 9009a^2 + 3(1001b^2 + 1404bc + 495c^2)\right]$
Gauss' model	$E = \dfrac{\pi^2 G\delta_0^2 R^5}{\beta^4}\left(\dfrac{2\sqrt{\pi}\exp(-\beta^2)\operatorname{erf}(\beta)}{\beta} - 2\exp(-2\beta^2) - \dfrac{\sqrt{2\pi}\operatorname{erf}(\sqrt{2}\beta)}{2\beta}\right)$

Table 4

Estimations of the gravitational potential energy *E* (spherical Earth)

Model	E, ergs
Mescheryakov, 1973	-2.34×10^{39}
Rubincam, 1979	-2.45×10^{39}

Homogeneous planet	-2.2419×10^{39}
Legendre-Laplace law	-2.4595×10^{39}
Roche's law	-2.4802×10^{39}
Bullard's model	-2.4716×10^{39}
Gauss' model	-2.5009×10^{39}

Table 5

E-estimates for the spherical Earth with one density-jump at the core/mantle boundary

Model (2 shells)	E, ergs	R.m.s. deviation from PREM in the core-mantle area, g/cm^3
Rubincam, 1979	$-2.45\ \ \times10^{39}$	–
Legendre-Laplace law	-2.4944×10^{39}	0.430
Roche's law	-2.4938×10^{39}	0.409
Bullard's model	-2.4907×10^{39}	0.322
Gauss' model	-2.4940×10^{39}	0.437

It has to be pointed out, that the energy E derived by Mescheryakov (1973) as $2E = -V_m M$ was based on the known Earth's mass M and the mean-value theorem after preliminary computation of the mean value V_m of the internal potential V_i in Eq. (8). The estimation of E given by Rubincam (1979) was found for the spherical Earth differentiated into homogeneous core and homogeneous mantle with one jump at the core-mantle boundary. We will apply a similar approach to the above-discussed profiles using the direct approximation of the PREM density by these four simplest piecewise models separated into two shells with the same basic jump at the core-mantle boundary. Table 5 illustrates results of such approximation in the form of r.m.s. deviations from the PREM density based in every case on the additional conditions to keep δ_m, I_m, and δ_s. Despite the best value of r.m.s. for the Bullard's model we prefer to use below a simpler Roche's law due to a smaller number of the parameters a_j and b_j (j=1,2,…k) introduced for each shell.

The comparison of E-values from Table 4 and Table 5 gives better-quality agreement between all values of E when the basic jump of density at the core/mantle boundary is taking into consideration. E-estimates given in Table 5 satisfy again to the inequality (9) with the two limits E_{Gauss} and E_{H} from Table 4. All values of E in the case of these piecewise models from Table 5 are very close to the minimum amount E_{Gauss}. For this reason the accuracy σ^E_{Gauss} of E_{Gauss} was derived under the assumption that σ^E_{Gauss} depends only on accuracy of δ_m and I_m given above. Numerically we get $E_{\text{Gauss}} = (-2.5009 \pm 0.0025)\times10^{39}$ ergs. Hence, if a spherical Earth differentiates into present-day core and mantle we get in view of the estimated accuracy $\sigma^E_{\text{Gauss}} = \pm0.0025\times10^{39}$ ergs a perfect accordance between E-values corresponded to the layered Legendre-Laplace, Roche, Bullard, and Gauss models with 2 shells. This quantity σ^E_{Gauss} is certainly larger than E-estimates contained in the 2nd-degree harmonics (Rubincam, 1979) and for this reason we will use again radial-only piecewise model for the determination of the potential energy E of the ellipsoidal Earth.

The internal potential V_i inside the ellipsoid of revolution with the radial density $\delta(r = \rho \cdot R)$ was adopted according to Moritz (1990, p.41). For the homothetic stratification f=const we get

$$V_i = V_i^{sphera} + \Delta V_i^{ell} = \frac{4\pi G}{r}\int_0^r \delta(r) r^2 dr + 4\pi G \int_r^R \delta(r) r dr + \Delta V_i^{ell}, \tag{10}$$

$$\Delta V_i^{ell} = \frac{8\pi G \cdot f}{3r} P_2(\cos\vartheta) \int_0^r \delta(r) r^2 dr - \frac{8\pi G \cdot f}{3r} P_2(\cos\vartheta) \int_0^r \delta(r) r^4 dr, \tag{11}$$

the internal potential of the ellipsoidal Earth [Eq. (10)] in the form of the internal potential of the spherical planet V_i^{sphere} reduced to V_i by the ellipsoidal reduction ΔV_i^{ell} [Eq. (11)]. Eqs. (10 – 11) allow the direct computation of the potential energy E in the following way

$$E = E_{Sphere} + \Delta E_{ell}, \tag{12}$$

if inserted into Eq. (8). Then, taking into account the flattening *f* we will determine the ellipsoidal reduction ΔE_{ell} beforehand. Since the values *E* of the piecewise radial models with one jump (Table 5) are very close to the lower limit E_{Gauss} in Eq. (9) it is enough to estimate ΔE_{ell} by applying the Gauss' continuous model inside the ellipsoid with the homothetic stratification. Numerically we get $\Delta E_{ell}^{\mathrm{Gauss}} \approx 0.000045\times10^{39}$ ergs two orders smaller value than accuracy $\sigma_{\mathrm{Gauss}}^{E} = \pm 0.0025\times10^{39}$ ergs. Hence, it is sufficient to adopt the reduction $\Delta V_i^{ell} = 0$ in Eq. (10) for the internal potential V_i.

If the expression for *E* is known, the piecewise PREM profile is one of a most suitable densities for the estimation of the potential energy *E*, although this problem is not discussed in the literature. Due to polynomials of different powers adopted for each shell there are significant difficulties in the derivation of such relationship for *E* in the case of the PREM model. Therefore we will apply another appropriate model represented by polynomials of identical even powers within every shell. Because the PREM-profile agrees well with the piecewise Roche model (Marchenko, 2000) consisting from 7 shells (Fig. 4), we use this Roche density as initial information in the following form

$$\delta_j(r) = a_j + b_j\left(\frac{r}{R}\right)^2, \qquad a_0 = b_0 = 0, \tag{13}$$

where j=0,1,2,…k, k is the number of shells (k=7), a_j and b_j are the known coefficients of the model (13) given for each shell separately (Table 6) with the artificial zero shell $a_0 = b_0 = 0$ involved here for the generalization of basic formulae. Note also that r.m.s. deviation between these models (Fig. 4) has the value 0.06 g/cm^3 for the most important in our case core-mantle area and increases only to 0.24 g/cm^3 for the total Earth (core-mantle-crust).

Table 6

Piecewise Roche's model with 7 basic shells as sampled for the PREM (Marchenko, 2000)

j (Shell)	a_j, g/cm^3	b_j, g/cm^3	r_j, km
1 (Inner core)	13.061	-8.891	1221.5
2 (Outer core)	12.483	-8.343	3480.0
3 (Lower mantle)	6.370	-2.574	5701.0
4 (Upper mantle)	6.058	-2.577	5971.0
5 (Upper mantle)	5.784	-2.524	6151.0
6 (Upper mantle)	6.057	-2.903	6346.6
7 (Crust)	6.622	-3.952	

With $\Delta V_i^{ell} = 0$, r_0=0, and a current point lied within the *j* shell at the distance *r*, the substitution of Eq. (16) into Eq. (13) provides the expression for the internal potential

$$V_i(r) = \frac{4\pi G}{r}\sum_{l=1}^{j}\int_{r_{l-1}}^{r_l}\delta_l(r)r^2dr + \frac{4\pi G}{r}\int_{r_j}^{r}\delta_j(r)r^2dr + 4\pi G\int_{r}^{r_{j+1}}\delta_{j+1}(r)rdr + 4\pi G\sum_{l=j+1}^{k}\int_{r_l}^{r_{l+1}}\delta_l(r)rdr\,. \tag{14}$$

Then we substitute Eq. (13) into Eq. (14), obtaining

$$V_i(r) = \frac{GM_j}{r} + \frac{4\pi G}{3r}\left[a_j(r^3 - r_j^3) + \frac{3b_j}{5R^2}(r^5 - r_j^5)\right] + \frac{4\pi G}{2}\left[a_j(r_{j+1}^2 - r^2) + \frac{b_j}{2R^2}(r_{j+1}^4 - r^4)\right] + C_{\mathrm{int}}^j\,, \tag{15}$$

$$\left.\begin{aligned} M_j &= \frac{4\pi}{3}\sum_{l=1}^{j}\left[a_l(r_l^3 - r_{l-1}^3) + \frac{3b_l}{5R^2}(r_l^5 - r_{l-1}^5)\right],\\ C_{\mathrm{int}}^j &= \frac{4\pi G}{2}\sum_{l=j+1}^{k}\left[a_l(r_{l+1}^2 - r_l^2) + \frac{b_l}{2R^2}(r_{l+1}^4 - r_l^4)\right]. \end{aligned}\right\} \tag{16}$$

Table 7

Estimations of the potential energy *E* derived from the piecewise Roche's model (Table 6)

j (Shell)	Contribution E_j of each shell, ergs
1 (Inner core)	-0.0541×10^{39}
2 (Outer core)	-0.9159×10^{39}
3 (Lower mantle)	-1.1625×10^{39}

4 (Upper mantle)	-0.1527×10^{39}
5 (Upper mantle)	-0.0954×10^{39}
6 (Upper mantle)	-0.0998×10^{39}
7 (Crust)	-0.0104×10^{39}
Total gravitational potential energy: -2.4910×10^{39} ergs	

Thus, according to Eqs. (14 – 15) in the case of the piecewise Roche's density (13) the internal potential $V_i(r)$ at the arbitrary current point *P* can be formed from the four parts: first two terms and last two terms in Eq. (18) represent the potentials of the Earth's layers lied below and above *P*, respectively. By this, after some algebraic manipulation with Eq. (15) inserted into Eq. (8) we get a simple possibility of the determination of the gravitational potential energy *E*. The result is

$$\left.\begin{aligned} E &= -2\pi\sum_{j=1}^{k}\int_{r_{j-1}}^{r_j}\delta_j(r)V_i(r)r^2dr = \sum_{j=1}^{k}E_j,\\ E_j &= -2\pi\sum_{m=1}^{8}c_{mj}, \end{aligned}\right\} \tag{17}$$

where E_j expresses the contribution of each *j*-shell in the total value *E* by

$$\left.\begin{aligned} c_{1j} &= a_jM_j(r_j^2 - r_{j-1}^2)/2,\\ c_{2j} &= 4\pi Ga_j(a_jA_j + b_jB_j),\\ c_{3j} &= 4\pi Ga_j(a_jA_j + b_jC_j),\\ c_{4j} &= a_jC_{\text{int}}^j(r_j^3 - r_{j-1}^3)/3,\\ c_{5j} &= b_jM_j(r_j^4 - r_{j-1}^4)/(4R^2),\\ c_{6j} &= 4\pi Gb_j(a_jC_j + b_jD_j/R^2),\\ c_{7j} &= 4\pi Gb_j(a_jB_j + b_jD_j/R^2),\\ c_{8j} &= b_jC_{\text{int}}^j(r_j^5 - r_{j-1}^5)/(5R^2), \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} A_j &= (3r_{j-1}^5 - 5r_{j-1}^3r_j^2 + 2r_j^5)/30,\\ B_j &= (5r_{j-1}^7 - 7r_{j-1}^5r_j^2 + 2r_j^7)/(70R^2),\\ C_j &= (3r_{j-1}^7 - 7r_{j-1}^3r_j^4 + 4r_j^7)/(84R^2),\\ D_j &= (5r_{j-1}^9 - 9r_{j-1}^5r_j^4 + 4r_j^9)/(180R^2). \end{aligned}\right\} \tag{19}$$

With adopted piecewise Roche's density (Fig. 4, Table 6) we get the estimation of the potential energy *E* (Table 7) by means of the contributions E_j of each shell [Eq. (17)]. The obtained quantity $E = -2.4910\times10^{39}$ ergs agrees well with *E*-estimates from Table 5 based on the radial profiles with one jump at the core-mantle boundary and satisfies to the inequality (9) at the vicinity $E_{\text{Gauss}} = (-2.5009 \pm 0.0025)\times10^{39}$ ergs of the minimum amount. In view of the accuracy $\sigma_{\text{Gauss}}^E = \pm0.0025\times10^{39}$ ergs we get a remarkable accordance between $E = -2.4910\times10^{39}$ ergs derived from the piecewise Roche's density with 7 basic shells as sampled for PREM and the values *E* given by the simplest piecewise Legendre-Laplace, Roche, Bullard, and Gauss models with 2 shells all corresponded to the spherically symmetric Earth differentiated into core and mantle only. We may assume that the quantity $E = -2.4910\times10^{39}$ ergs will be close to *E*-value of the PREM model, taking into account a minimum contribution E_7 of the crust (Table 7) into the total *E* and r.m.s. deviation between PREM and piecewise Roche's models.

4 Conclusions. The global density model inside the ellipsoidal Earth was adopted as exact solution of the restricted three-dimensional Cartesian moments problem for $\delta(\rho,\vartheta,\lambda)$ under the conditions to conserve the Earth's mass, the geometrical flattening, and all principal moments of inertia. This model includes the reference radial density profile $\delta(\rho)_{\text{R}}$ selected in the form of the piecewise Roche's models with 7 basic shells, taking into account density jumps as sampled for PREM. With $\delta(\rho)_{\text{R}}$ chosen as exact constituent, the accuracy $\sigma_{\delta(\rho,\vartheta,\lambda)}$ of continuous global density was derived from the consistent set of the Earth's mechanical parameters.

Comparison of the lateral density anomalies $\Delta\delta(\rho,\vartheta,\lambda)$ with the accuracy $\sigma_{\delta(\rho,\vartheta,\lambda)}$ at the same depths leads generally at least to values of the same order in uncertainties and density heterogeneities.

As a result, only radial density models were adopted for the determination of the gravitational potential energy *E*. Relationships for *E* were derived in the following cases: 1) continuous Legendre-Laplace, Roche, Bullard, and Gauss radial density laws; 2) the same radial models with one added jump of density at the core-mantle boundary (2 shells); 3) the piecewise Roche's profile separated into 7 shells. The estimation of *E* according to various continuous density laws gives the following result: there are two limits for all computed E. First one agrees with the homogeneous distribution. Second one corresponds to the Gauss' radial density.

Finally all determinations of the potential energy *E* were made for the spherical Earth since the ellipsoidal reduction ΔE_{ell} gives two orders smaller quantity than the estimated accuracy $\sigma^{E}_{\text{Gauss}} = \pm 0.0025 \times 10^{39}\ \text{ergs}$. Taking into account $\sigma^{E}_{\text{Gauss}}$ we get a perfect agreement between $E_{\text{Gauss}} = (-2.5009 \pm 0.0025) \times 10^{39}\ \text{ergs}$, the potential energy $E = -2.4910 \times 10^{39}\ \text{ergs}$ derived from the piecewise Roche's density with 7 basic shells, and the values *E* given by the four simplest piecewise Legendre-Laplace, Roche, Bullard, and Gauss models with 2 shells (core and mantle only).

Among continuous densities for the Earth's interior given in Fig. 3 the Gaussian distribution (based on the Earth's fundamental parameters δ_m, I_m, δ_s only) allows a better-quality representation of the general trend of the planet's piecewise density. By this, the Gauss' model leads to the reliable estimation of the lower limit E_{Gauss} of the potential energy *E*, answering in this manner on the question above about the gravitational potential energy *E* of the 'true' Earth (Moritz, 1990): all piecewise density models give *E*-values at the vicinity $E_{\text{Gauss}} = -2.5009 \times 10^{39}\ \text{ergs}$ of the lower limit of Eq. (9).

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