REACHABILITY AND OBSERVABILITY OF FRACTIONAL POSITIVE ELECTRICAL CIRCUITS

Tadeusz Kaczorek

Białystok University of Technology, Białystok, Poland,
kaczorek@isep.pw.edu.pl

Abstract: Necessary and sufficient conditions for the reachability and observability of fractional positive continuous-time linear electrical circuits are established. Effectiveness of the proposed conditions is demonstrated on examples of electrical circuits.

Key words: reachability, observability, fractional, positive, electrical circuits

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [5, 9]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [22, 23]. These notions are the basic concepts of the modern control theory [1, 21, 27, 30, 31]. They have been also extended to positive linear systems [5, 9]. The decomposition of the pair (*A*,*B*) and (*A*,*C*) of the positive discrete-time linear system has been addressed in [7]. The positive circuits and their reachability has been investigated in [12] and controllability and observability of electrical circuits in [10].

The reachability of linear systems is closely related to the controllability of the systems. Specially for positive linear systems, the conditions for the controllability are much stronger than for the reachability [9]. Tests for the reachability and controllability of standard and positive linear systems are given in [9, 16]. The positivity and reachability of fractional electrical circuits have been addressed in [12, 13]. The finite zeros of positive discrete-time and continuous-time linear systems have been investigated in [28-30] and the decoupling zeros of positive discrete-time linear systems in [11].

Mathematical fundamentals of fractional calculus are given in the monographs [24-26]. The positive fractional linear systems have been introduced in [8,]. Stability of fractional linear 1D discrete–time and continuous-time systems has been investigated in the papers [2, 4, 17, 18], and that of 2D fractional positive linear systems in [6]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [15], and the positive linear systems consisting of *n* subsystems with different fractional order has been analyzed in [14]. Some recent interesting results in the fractional systems theory and its applications can be found in [3, 18, 24-26]. The reachability and observability of fractional positive continuous-time linear systems have been addressed in [20] and constructability and observability of standard and positive electrical circuits in [19].

In this paper, new necessary and sufficient conditions for the reachability and observability of fractional positive continuous-time linear electrical circuits will be established.

The paper is organized as follows. In section 2, some basic definitions and theorems concerning positive continuous-time linear systems are recalled. The fractional positive electrical circuits are addressed in section 3. The main result of the paper is given in sections 4 and 5. In section 4, new necessary and sufficient conditions for the reachability of fractional positive electrical circuits are established and for their observability - in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ is the set of real numbers, $\Re^{n \times m}$ represents the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ denotes the set of $n \times m$ matrices with nonnegative entries, and $\Re_+^n = \Re_+^{n \times 1}$, M_n stand for the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n \times n$ identity matrix.

2. Fractional positive linear continuous-time systems

The following Caputo definition of the fractional derivative will be used [18, 24, 25]

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (2.1)$$

$$n-1 < \alpha < n \in N = \{1, 2, \ldots\}$$

where $\alpha \in \Re$ is the order of fractional derivative and

$$f^{(n)}(\tau) = \frac{\mathrm{d}^\alpha f(\tau)}{\mathrm{d}\tau^\alpha} \quad \text{and} \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t \quad \text{is the}$$

gamma function.

Consider the continuous-time fractional linear system described by the state equations

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \qquad (2.2a)$$

$$y(t) = Cx(t) + Du(t), \qquad (2.2b)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}, B \in \Re^{n \times m}$, $C \in \Re^{p \times n}, D \in \Re^{p \times m}$.

Theorem 2.1. [18] The solution to equation (2.2a) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau,$$

$$x(0) = x_0$$

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \qquad (2.4)$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \qquad (2.5)$$

and $E_\alpha(At^\alpha)$ is the Mittag-Leffler matrix function.

Definition 2.1. [18] Fractional system (2.2) is called an internally positive fractional system if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m, t \geq 0$.

Theorem 2.2. [18] Continuous-time fractional system (2.2) is internally positive if and only if the matrix A is a Metzler matrix and

$$A \in M_n, B \in \Re_+^{n \times m}, C \in \Re_+^{p \times n}, D \in \Re_+^{p \times m}. \quad (2.6)$$

Lemma 2.1. The Mittag-Leffler matrix function (2.4) satisfies the equation

$$\frac{\mathrm{d}^\alpha \Phi_0(t)}{\mathrm{d}t^\alpha} = A\Phi_0(t). \qquad (2.7)$$

Proof. From (2.2a) and (2.3) for $Bu(t) = 0$ we have

$$\frac{\mathrm{d}^\alpha \Phi_0(t)x_0}{\mathrm{d}t^\alpha} = A\Phi_0(t)x_0. \qquad (2.8)$$

Therefore, Equality (2.7) holds since equation (2.8) is satisfied for arbitrary $x_0 \neq 0$.

3. Fractional positive electrical circuits

Let the current $i_C(t)$ in a supercondensator (shortly condensator) with the capacity C be the α order derivative of its charge $q(t)$ [13, 19]

$$i_C(t) = \frac{\mathrm{d}^\alpha q(t)}{\mathrm{d}t^\alpha}, 0 < \alpha < 1. \qquad (3.1)$$

Using $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{\mathrm{d}^\alpha u_C(t)}{\mathrm{d}t^\alpha} \qquad (3.2)$$

where $u_C(t)$ is the voltage on the condensator.

Similarly, let the voltage $u_L(t)$ on a coil (inductor) with the inductance L be the β order derivative of its magnetic flux $\Psi(t)$ [13, 19]

$$u_L(t) = \frac{\mathrm{d}^\beta \Psi(t)}{\mathrm{d}t^\beta}, 0 < \beta < 1, \qquad (3.3)$$

Taking into account that $\Psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{\mathrm{d}^\beta i_L(t)}{\mathrm{d}t^\beta}, \qquad (3.4)$$

where $i_L(t)$ is the current in the coil.

Consider an electrical circuit composed of resistors, n capacitors and m voltage sources. Using equation (3.2) and Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), 0 < \alpha < 1, \qquad (3.5)$$

where $x(t) \in \Re^n, u(t) \in \Re^m, A \in \Re^{n \times n}, B \in \Re^{n \times m}$, The components of the state vector $x(t)$ and input vector $u(t)$ are the voltages on the condensators and source voltages respectively.

Similarly, using equation (3.4) and Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{\mathrm{d}^\beta x(t)}{\mathrm{d}t^\beta} = Ax(t) + Bu(t), 0 < \beta < 1, \qquad (3.6)$$

where $x(t) \in \Re^n, u(t) \in \Re^m, A \in \Re^{n \times n}, B \in \Re^{n \times m}$. In this case, the components of the state vector $x(t)$ are the currents in the coils.

The solution to equation (3.5) (or (3.6)) satisfying the initial condition $x(0) = x_0$ is given by (2.3).

Now let us consider an electrical circuit composed of resistors, capacitors, coils and a voltage (current) source. As the state variables (the components of the state vector $x(t)$) we choose the voltages on the capacitors and the currents in the coils. Using equations (3.2), (3.4) and Kirchhoff's laws, we

may write for the fractional linear circuits in the transient states the state equation

$$\begin{bmatrix} \dfrac{\mathrm{d}^\alpha x_C}{\mathrm{d}t^\alpha} \\ \dfrac{\mathrm{d}^\beta x_L}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \ 0 < \alpha, \beta < 1, \ (3.7a)$$

where the components $x_C \in \mathfrak{R}^{n_1}$ are the voltages on the condensators, the components $x_L \in \mathfrak{R}^{n_2}$ are the currents in the coils and the components of $u \in \mathfrak{R}^m$ are the source voltages and

$$A_{ij} \in \mathfrak{R}^{n_i \times n_j}, B_i \in \mathfrak{R}^{n_i \times m}, i, j = 1, 2. \quad (3.7b)$$

Theorem 3.1. The solution to equation (3.7) for $0 < \alpha < 1; 0 < \beta < 1$ with the initial conditions

$$x_C(0) = x_{10} \text{ and } x_L(0) = x_{20} \quad (3.8)$$

has the form

$$x(t) = \Phi_0(t)x_0 +$$
$$+ \int_0^t \left[\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01} \right] u(\tau)\mathrm{d}\tau , \quad (3.9a)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$T_{kl} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for } k + l > 0 \end{cases}, \quad (3.9b)$$

$$\Phi_0(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + l\beta}}{\Gamma(k\alpha + l\beta + 1)},$$

$$\Phi_1(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha + l\beta - 1}}{\Gamma\left[(k+1)\alpha + l\beta\right]},$$

$$\Phi_2(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + (l+1)\beta - 1}}{\Gamma\left[k\alpha + (l+1)\beta\right]}.$$

The proof is given in [18].

The extension of Theorem 3.1 to the systems consisting of n subsystems with different fractional orders is given in [14].

Definition 3.1. The fractional electrical circuit (3.5) (or (3.6), (3.7)) is called a (an) (internally) positive fractional system if the state vector $x(t) \in \mathfrak{R}_+^n$, $t \geq 0$ for any initial conditions $x_0 \in \mathfrak{R}_+^n$ and all inputs $u(t) \in \mathfrak{R}_+^m, t \geq 0$.

Theorem 3.2. Fractional electrical circuit (3.5) (or (3.6)) is (internally) positive if and only if

$$A \in M_n, B \in \mathfrak{R}_+^{n \times m} \quad (3.10)$$

where M_n is the set of $n \times n$ Metzler matrices.

The proof is given in [14, 18].

Theorem 3.2 applied to the fractional circuit (3.7) proves that the electrical circuit is positive if and only if

$$A_{kk} \in M_{n_k}, k = 1, 2 ;$$

$$A_{12} \in \mathfrak{R}_+^{n_1 \times n_2}, \ A_{21} \in \mathfrak{R}_+^{n_2 \times n_1},$$

$$B_1 \in \mathfrak{R}_+^{n_1 \times m}, B_2 \in \mathfrak{R}_+^{n_2 \times m} \quad (3.11)$$

Consider the fractional electrical circuit shown in Figure 3.1 with the given conductances $G_k, k = 0, 1, \ldots, n$; capacitances $C_j, j = 1, \ldots, n$ and source voltage e.
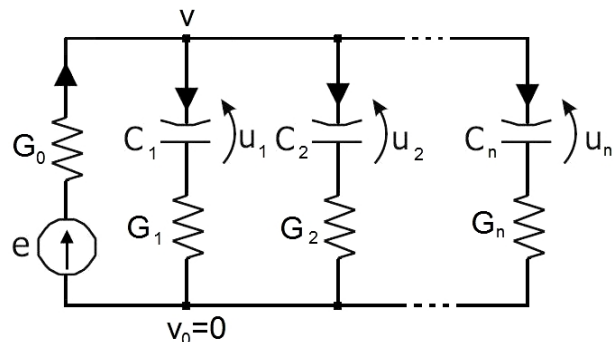

Fig. 3.1. Fractional electrical circuit.

Using (3.2) and Kirchhoff's laws, we may write the equations

$$C_k \frac{\mathrm{d}^\alpha u_k}{\mathrm{d}t^\alpha} = G_k(v - u_k), k = 1, \ldots, n \quad (3.12)$$

and

$$G_0(e - v) = \sum_{j=1}^n G_j(v - u_j). \quad (3.13)$$

From (3.13) we have

$$v = \frac{1}{G}\left(G_0 e + \sum_{j=1}^n G_j u_j \right), G = \sum_{i=0}^n G_i. \quad (3.14)$$

Substitution of (3.14) into (3.12) yields

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + Be, \quad (3.15a)$$

where

$$A = \begin{bmatrix} -\dfrac{G_1 G - G_1^2}{C_1 G} & \dfrac{G_1 G_2}{C_1 G} & \cdots & \dfrac{G_1 G_n}{C_1 G} \\[2mm] \dfrac{G_2 G_1}{C_2 G} & -\dfrac{G_2 G - G_2^2}{C_2 G} & \cdots & \dfrac{G_2 G_n}{C_2 G} \\[2mm] \vdots & \vdots & \ddots & \vdots \\[2mm] \dfrac{G_n G_1}{C_n G} & \dfrac{G_n G_2}{C_n G} & \cdots & -\dfrac{G_n G - G_n^2}{C_n G} \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{G_0 G_1}{C_1 G} \\[2mm] \vdots \\[2mm] \dfrac{G_0 G_n}{C_n G} \end{bmatrix} \qquad (3.15\text{b})$$

From (3.15b) we can see that $A \in M_n$, and $B \in \Re_+^n$. Therefore, the following theorem has been proved.

Theorem 3.3. The fractional electrical circuit shown in Fig. 3.1 is positive for any values of the conductances G_k, $k = 0,1,...,n$; capacitances C_j, $j = 1,...,n$ and source voltage e.

Consider the electrical circuit shown in Fig. 3.2 with given resistances $R_k, k = 1,...,n$, inductances $L_2, L_4,...,L_{n_2}$, capacitances $C_1, C_3,...,C_{n_1}$ and source voltages $e_1, e_2,...,e_n$.
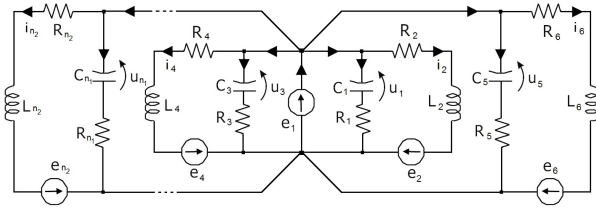

Fig. 3.2. Fractional electrical circuit.

Using Kirchhoff's laws, we can write the equations

$$e_1 = R_k C_k \frac{d^\alpha u_k}{dt^\alpha} + u_k \quad \text{for} \quad k = 1,3,...,n_1, (3.16\text{a})$$

$$e_1 + e_j = L_j \frac{d^\beta i_j}{dt^\beta} + R_j i_j \text{ for } j = 2,4,...,n_2, (3.16\text{b})$$

which can be written in the form

$$\begin{bmatrix} \dfrac{d^\alpha u}{dt^\alpha} \\[2mm] \dfrac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + Be , \qquad (3.17\text{a})$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix}, \quad u = \begin{bmatrix} e_1 \\ e_2 \\ e_4 \\ \vdots \\ e_n \end{bmatrix}, \quad (n = n_1 + n_2) \ (3.17\text{b})$$

and

$$A = \operatorname{diag}\left[-\frac{1}{R_1 C_1}, -\frac{1}{R_3 C_3}, ..., - \right.$$

$$\left. -\frac{1}{R_{n_1} C_{n_1}}, -\frac{R_2}{L_2}, -\frac{R_4}{L_4}, ..., -\frac{R_{n_2}}{L_{n_2}} \right] \in M_n,$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n \times (n_1 + n_2)},$$

$$B_1 = \begin{bmatrix} \dfrac{1}{R_1 C_1} & 0 & 0 & \cdots & 0 \\[2mm] \dfrac{1}{R_3 C_3} & 0 & 0 & \cdots & 0 \\[2mm] \vdots & \vdots & \vdots & \cdots & \vdots \\[2mm] \dfrac{1}{R_{n_1} C_{n_1}} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \dfrac{1}{L_2} & \dfrac{1}{L_2} & 0 & \cdots & 0 \\[2mm] \dfrac{1}{L_4} & 0 & \dfrac{1}{L_4} & \cdots & 0 \\[2mm] \vdots & \vdots & \vdots & \ddots & \vdots \\[2mm] \dfrac{1}{L_{n_2}} & 0 & 0 & \cdots & \dfrac{1}{L_{n_2}} \end{bmatrix}. \ (3.17\text{c})$$

The electricфд circuit described by equation (3.17) is positive for all value of the resistances R_k, $k = 1,...,n_1 + n_2$ inductances L_k, $k = 2,4,...,n_2$, capacitances C_k, $k = 1,3,...,n_1$. Therefore, the following theorem has been proved.

Theorem 3.4. The fractional linear electrical circuit of the structure shown in Fig. 3.2 is positive for any values of its resistances, inductances and capacitances.

The positivity and reachability of the fractional electrical circuits have been investigated in [13].

4. Reachability

Consider the fractional positive electrical circuits described by equation (3.7a).

Definition 4.1. Fractional positive electrical circuits (3.7a) is called reachable in time $\left[0, t_f \right]$ if for every

given final state $x_f = \begin{bmatrix} x_{Cf} \\ x_{Lf} \end{bmatrix} \in \Re_+^n$ there exists an input

$u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state $\begin{bmatrix} x_C \\ x_L \end{bmatrix}$ of

the circuit from the zero initial state $x_0 = \begin{bmatrix} x_{C0} \\ x_{L0} \end{bmatrix}$ to the

final step x_f.

A real square matrix is called monomial if each row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 4.1. Fractional positive electrical circuit (3.7a) is reachable in time $[0, t_f]$ if and only if the matrix

$$R_f = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) d\tau \in \Re_+^{n \times n} \quad (4.1a)$$

is monomial, where

$$\Phi(\tau) B = \Phi_1(\tau) B_{10} + \Phi_2(\tau) B_{01}, \quad (4.1b)$$

where $\Phi_1(\tau), \Phi_2(\tau)$ and B_{10}, B_{01}, are defined by (3.9b).

Proof. Note that $R_f^{-1} \in \Re_+^{n \times n}$ if and only if the matrix (4.1a) is monomial. The input

$$u(t) = B^T \Phi^T(t_f - t) R_f^{-1} x_f \quad (4.2)$$

steers the state of electrical circuit (3.7a) from $x_0 = 0$ to $x_f \in \Re_+^n$, since using (3.9) for $x_0 = 0$ and (4.2), we obtain

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B u(\tau) d\tau =$$

$$= \int_0^{t_f} \Phi(t_f - \tau) B B^T \Phi^T(t_f - \tau) d\tau R_f^{-1} x_f =$$

$$= \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) d\tau R_f^{-1} x_f = x_f. \quad (4.3)$$

Example 4.1. Consider the fractional electrical circuit shown in Fig. 4.1 with given resistances R_1, R_2, R_3; inductances L_1, L_2, and source voltages e_1, e_2. Give the condition under which the electrical circuit is reachable in time $[0, t_f]$ and find the input $u(t) \in \Re_+^2$, $t \geq 0$ which steers the system from $x_0 = 0$ to the final state x_f.

Using Kirchhoff's laws, we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{d^\beta i_1}{dt^\beta},$$
$$\quad 0 < \beta < 1, \quad (4.4)$$
$$e_2 = R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{d^\beta i_2}{dt^\beta},$$

which can be written in the form

$$\frac{d^\beta}{dt^\beta} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (4.5)$$

where

$$A = \begin{bmatrix} -\dfrac{R_1 + R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}. \quad (4.6)$$
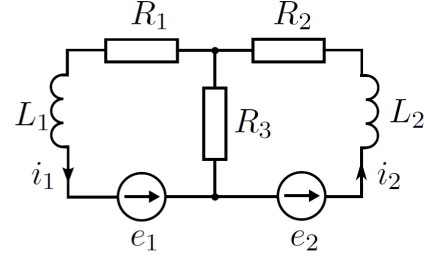

Fig. 4.1. Fractional electrical circuit.

The electrical circuit is positive since the matrix A is a Metzler matrix and the matrix B has nonnegative entries.

We shall show that the positive electrical circuit is reachable if and only if $R_3 = 0$. If $R_3 = 0$, then

$$A = \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 \\ 0 & -\dfrac{R_2}{L_2} \end{bmatrix} \quad (4.7)$$

and

$$\hat{O}(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\beta - 1}}{\Gamma[(k+1)\beta]} =$$

$$= \sum_{k=0}^\infty \frac{t^{(k+1)\beta-1}}{\Gamma[(k+1)\beta]} \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 \\ 0 & -\dfrac{R_2}{L_2} \end{bmatrix}^k \quad (4.8)$$

and from (4.1) we obtain

$$R_f = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) d\tau =$$

$$= \int_0^{t_f} \begin{bmatrix} \dfrac{1}{L_1^2} & 0 \\ 0 & \dfrac{1}{L_2^2} \end{bmatrix} \Phi^2(\tau) d\tau \quad (4.9)$$

Matrix (4.9) is monomial and by Theorem 4.1 the fractional positive electrical circuit is reachable if and only if $R_3 = 0$, since for $R_3 \neq 0$ the matrix R_f is not monomial.

The desired input which steers the state
$x(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$ from $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final state

$x_f \in \Re_+^2$ is given by

$$u(t) = B^T \Phi^T (t_f - t) R_f^{-1} x_f =$$

$$= \sum_{k=0}^{\infty} \frac{(t_f - t)^{(k+1)\beta - 1}}{\Gamma[(k+1)\beta]} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}^k R_f^{-1} x_f \qquad (4.10)$$

Example 4.2. Consider the fractional electrical circuit shown in Fig. 4.2 with given conductances G_1, G_1', G_2, G_2', G_{12}; capacitances C_1, C_2 and source voltages e_1, e_2.
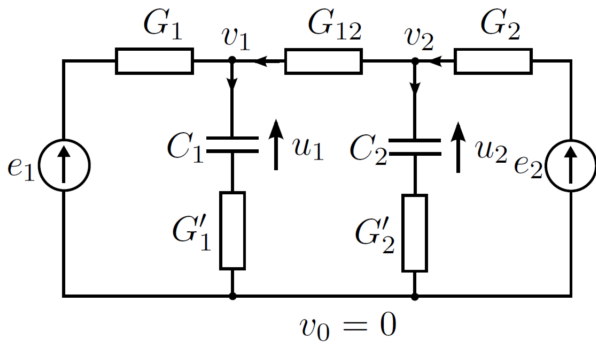

Fig. 4.2. Fractional electrical circuit.

Using Kirchhoff's laws, we can write the equations

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$0 < \alpha < 1 \qquad (4.11)$$

and

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$$

$$= -\begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \qquad (4.12a)$$

where

$$G_{11} = G_1 + G_1' + G_{12}, \; G_{22} = G_2 + G_2' + G_{12}. \qquad (4.12b)$$

Taking into account that the matrix

$$\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix} \qquad (4.13)$$

is nonsingular and

$$-\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \in \Re_+^{2 \times 2} \qquad (4.14)$$

from (4.12) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \cdot$$

$$\cdot \left\{ \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\} \qquad (4.15)$$

Substitution of (4.15) into (4.11) yields

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \qquad (4.16a)$$

where

$$A = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \cdot$$

$$\cdot \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} - \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \in M_2,$$

$$B = -\begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} -G_{11} & G_{12} \\ G_{12} & -G_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \in \Re_+^{2 \times 2}. \qquad (4.16b)$$

From (4.16b) we can see that A is the Metzler matrix and the matrix B has nonnegative entries. Therefore, the fractional electrical circuit is positive for all values of conductances and capacitances.

We shall show that the fractional positive electrical circuit is reachable in time $[0, t_f]$ if and only if $G_{12} = 0$.

If $G_{12} = 0$, then matrices (4.16b) are diagonal of the form

$$A = \begin{bmatrix} -\dfrac{G_1 G_1'}{C_1(G_1 + G_1')} & 0 \\ 0 & -\dfrac{G_2 G_2'}{C_2(G_2 + G_2')} \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{G_1 G_1'}{C_1(G_1 + G_1')} & 0 \\ 0 & \dfrac{G_2 G_2'}{C_2(G_2 + G_2')} \end{bmatrix} \qquad (4.17)$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \tag{4.18}$$

$$= \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \begin{bmatrix} -\dfrac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & -\dfrac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix}^k .$$

From (4.1) we obtain

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)d\tau = .$$

$$= \int_0^{t_f} \begin{bmatrix} \dfrac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & \dfrac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix}^2 \Phi^2(\tau)d\tau \tag{4.19}$$

From Theorem 4.1, the fractional positive electrical circuit is reachable if and only if $G_{12}=0$, since the matrix R_f is monomial only if $G_{12}=0$.

The desired input which steers the state

$$x(t) = u\begin{bmatrix} u_1(t) \\ i_2(t) \end{bmatrix} \quad \text{from} \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{to the final state}$$

$x_f \in \Re_+^2$ is given by

$$u(t) = B^T\Phi^T(t_f-t)R_f^{-1}x_f$$

$$= \sum_{k=0}^{\infty} \frac{(t_f-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} - \tag{4.20}$$

$$- \begin{bmatrix} -\dfrac{G_1 G_1'}{C_1(G_1+G_1')} & 0 \\ 0 & -\dfrac{G_2 G_2'}{C_2(G_2+G_2')} \end{bmatrix}^{k+1} R_f^{-1}x_f .$$

5. Observability

Consider the fractional positive electrical circuits described by the equations

$$\begin{bmatrix} \dfrac{d^\alpha x_C(t)}{dt^\alpha} \\ \dfrac{d^\beta x_L(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix} = Ax(t), \tag{5.1a}$$

$$y(t) = Cx(t), \tag{5.1b}$$

where

$$x(t) = \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \Re^{n\times n}, C \in \Re^{p\times n} \tag{5.1c}$$

and $y(t) \in \Re_+^p$ is the output of the electrical circuit.

It is well-known that [10] fractional electrical circuit (5.1) is positive if and only if

$$A \in M_n \quad \text{and} \quad C \in \Re_+^{p\times n}. \tag{5.2}$$

Definition 5.1. Fractional positive electrical circuit (5.1) is called observable in time $\begin{bmatrix} 0, t_f \end{bmatrix}$ if (knowing) the output $y(t) \in \Re_+^p$ for $t \in \begin{bmatrix} 0, t_f \end{bmatrix}$ is known, it is possible to find uniquely the initial value $x_0 = x(0) \in \Re_+^n$ of the state vector $x(t) \in \Re_+^n$.

Theorem 5.1. Fractional positive electrical circuit (5.1) is observable in time $\begin{bmatrix} 0, t_f \end{bmatrix}$ if and only if the matrix

$$V_f = \int_0^{t_f} \Phi_0^T(t)C^TC\Phi_0(t)dt \tag{5.3}$$

is monomial, where $\Phi_0(t)$ is defined by (3.9b).

Proof. From (3.9a) for $u(t)=0$, $t \geq 0$ we have

$$x(t) = \Phi_0(t)x_0. \tag{5.4}$$

Substitution of (5.4) into (5.1b) yields

$$y(t) = C\Phi_0(t)x_0. \tag{5.5}$$

Premultiplying (5.5) by $\Phi_0^T(t)C^T$ and integrating the product in the interval $\begin{bmatrix} 0, t_f \end{bmatrix}$, we obtain

$$\int_0^{t_f} \Phi_0^T(t)C^Ty(t)dt =$$

$$= \int_0^{t_f} \Phi_0^T(t)C^TC\Phi_0(t)dtx_0 = V_fx_0. \tag{5.6}$$

From (5.6) we can find $x_0 \in \Re_+^n$ if and only if matrix (5.3) is monomial, and $V_f^{-1} \in \Re_+^{n\times n}$.

Having compared Theorem 4.1 and 5.1, we have the following remark.

Remark 5.1. The conditions for observability of the fractional positive electrical circuits are dual to the reachability ones. By substituting in the reachability conditions $\Phi(t)$ and B by $\Phi_0^T(t)$ and C^T, we obtain the observability conditions.

6. Concluding remarks

The reachability and observability of the fractional positive electrical circuits composed of resistors, capacitors, coils and voltage (current) sources have been

addressed. New necessary and sufficient conditions for the reachability and observability of the fractional positive electrical circuits have been proposed (Theorems 4.1 and 5.1). The conditions for the observability are dual to the conditions for the reachability (Remark 5.1). The effectiveness of the proposed conditions has been demonstrated on examples of electrical circuits.

These considerations can be extended to fractional positive discrete-time linear systems and to fractional positive 2D continuous-discrete linear systems.

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ДОСЯЖНІСТЬ І СПОСТЕРЕЖНІСТЬ ДРОБОВИХ ДОДАТНИХ ЕЛЕКТРИЧНИХ КІЛ

Тадеуш Качорек

Встановлено необхідні й достатні умови досяжності й спостережності дробових додатних неперервних у часі лінійних електричних кіл. Ефективність запропонованих умов продемонстровано на прикладах електричних кіл.

Tadeusz Kaczorek – D.Sc., Professor, born in 1932 in Poland, received his M.Sc., Ph.D. and D.Sc. degrees in Electrical Engineering from Warsaw University of Technology, Poland, in 1956, 1962 and 1964, respectively. In the period 1968 – 1969 he was the dean of Electrical Engineering Faculty and in the period 1970–1973 – the pro-rector of Warsaw University of Technology, Poland. Since 1971 he has been a professor and since 1974 a full professor at Warsaw University of Techno-logy, Poland. In 1986 he was elected a corresponding member and in 1996 full member of Polish Academy of Sciences. In the period 1988–1991 he was the director of the Research Centre of Polish Academy of Sciences in Rome. In June 1999 he was elected the full member of the Academy of Engineering in Poland. In May 2004 he was elected to the honorary member of the Hungarian Academy of Sciences. He was awarded by the University of Zielona Gora, Poland (2002) by the title doctor honoris causa, the Technical University of Lublin, Poland (2004), the Technical University of Szczecin, Poland (2004), Warsaw University of Technology, Poland (2004), Bialystok University of Technology, Poland (2008), Lodz University of Technology, Poland (2009), Opole University of Technology, Poland (2009), Poznan University of Technology, Poland (2011), and Rzeszow University of Technology, Poland (2012).

His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has initiated the research in the field of singular 2D, positive 2D linear systems and positive fractional 1D and 2D systems. He has published 25 books (7 in English) and over 1000 scientific papers.

He supervised 69 Ph.D. theses. More than 20 of these PhD students became professors in USA, UK and Japan. He is the editor-in-chief of the Bulletin of the Polish Academy of Sciences, Technical Sciences and the editorial member of about ten international journals.