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Finite element approximations in projection methods for solution of some Fredholm integral equation of the first kind

Polishchuk O.

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, 3-b Naukova Str., 79060, Lviv, Ukraine

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Approximation properties of B-splines and Lagrangian finite elements in Hilbert spaces of functions defined on surfaces in three-dimensional space are investigated. The conditions for the convergence of Galerkin and collocation methods for solution of the Fredholm integral equation of the first kind for the simple layer potential that is equivalent to the Dirichlet problem for Laplace equation in $\mathbb{R}^3$ are established. The estimation of the error of approximate solution of this problem, obtained by means of the potential theory methods, is determined.

Keywords: *potential, integral equation, well-posed solvability, B-spline, Lagrange interpolation, Galerkin method, collocation method, convergence.*

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1. Introduction

Many physical processes (e.g. diffusion, heat flux, electrostatic field, perfect fluid flow, elastic motion of solid bodies, groundwater flow, etc.) are modeled using boundary value problems for Laplace equation [1–3]. The powerful tools for solving such problems are potential theory methods, especially in the case of tired boundary surface or complex shape surface [4–6]. These methods are a convenient for calculating desired solution in small domains [7, 8]. In number of cases, application of potential theory methods requires solving Fredholm integral equation of the first kind. In particular, one of the cases is solving Dirichlet problem in the space of functions with normal derivative jump on crossing boundary surface using simple layer potential [9, 10]. When solving Neumann problem in the space of functions with jump on crossing boundary surface using double layer potential, we also proceed to integral equation of the first kind [11, 12]. The need to solve integral equations of the first kind also arises when the sum of simple and double layer potentials is used to solve the double-sided Dirichlet or Neumann problem [13] or double-sided Dirichlet–Neumann problem [14] in the space of functions that, same as their normal derivatives, have jump on crossing boundary surface. Many systems of integral equations for the simple and double layer potentials that are equivalent to mixed boundary value problems for Laplace equation, also contain integral equations of the first kind [15, 16].

In general, researches of projection methods convergence mainly focus on solving integral equations of the second kind [4, 6, 17]. Defining well-posed solvability conditions for integral equations of the first kind that are equivalent to boundary value problems for Laplace's equation in Hilbert spaces [18–20] allows us to use projection methods for numerical solution of such equations, thus avoiding resource-consuming regularization procedures [21–23]. For detailed review of numerical methods for solving integral equations, please see [2–4, 6]. In [24, 25], convergence conditions are defined for the series of projection methods for solving Fredholm integral equation of the first kind for simple layer potential that is equivalent to three-dimensional Dirichlet problem for Laplace equation while approximating desired potential density with complete systems of orthonormal functions. However, if boundary surface has a complex shape usage of such approximations poses considerable difficulties for practical implementation of numerical methods [7]. In this case, finite elements of different types should be used for approximation of desired potential densities [26, 27]. Derived approximations, among other things, allow us to create effective algorithms for singularities removal in kernels and desired integral equation densities [28].

The purpose of the paper is to define convergence conditions of projection methods for approximate solution of Fredholm integral equations of the first kind by the example of integral equation for the simple layer potential that is equivalent to Dirichlet problem for Laplace equation using approximation of desired potential density with systems of finite elements of different types and orders.

2. Approximations of Hilbert spaces and the basic convergence theorem

Consider the operator equation

$$Au = f, \quad u \in U, \quad f \in F, \tag{1}$$

where U and F are the Hilbert spaces, $A \in L(U, F)$. To solve equation (1) we apply the projection method

$$Q_N A P_N u = Q_N f. \tag{2}$$

In formula (2) P_N and Q_N are projection operators from U and F onto closed finite dimensional subspaces $U_N \in U$ and $F_N \in F$ accordingly. Define operators P_N and Q_N in the following way. Denote by r_N the restriction operator from the space U to the finite dimensional subspace $V_N \subset \mathbb{R}^N$ and introduce in V_N the extension operator p_N as an isomorphism from V_N onto subspace $U_N \in U$. The norm in V_N is determined by the relation $\|\boldsymbol{u}_N\|_{V_N} = \|p_N \boldsymbol{u}_N\|_U$, $\boldsymbol{u}_N \in V_N$. Then $P_N = p_N r_N$ and we can determine the triple (V_N, p_N, r_N) as approximation of the space U. Such approximation are called convergent if

$$\lim_{N\to\infty} \|u - p_N r_N u\|_U = 0.$$

Denote by s_N the restriction operator from the space F to the finite dimensional subspace $\Phi_N \subset \mathbb{R}^N$. The extension operator q_N from Φ_N onto subspace $F_N = AU_N \subset F$ introduce by the formula

$$q_N \boldsymbol{f}_N = A p_N \boldsymbol{u}_N, \quad \boldsymbol{f}_N \in \Phi_N.$$

The norm in Φ_N is determined by the relation $\|\boldsymbol{f}_N\|_{\Phi_N} = \|q_N \boldsymbol{f}_N\|_F$, $\boldsymbol{f}_N \in \Phi_N$. Then $Q_N = q_N s_N$ and we can determine the triple (Φ_N, q_N, s_N) as approximation of the space F. Thus, the solution of problem (2) is reduced to solution of the system of linear algebraic equations

$$\boldsymbol{A}_N \boldsymbol{u}_N = \boldsymbol{f}_N, \quad \boldsymbol{A}_N = s_N A p_N, \quad \boldsymbol{A}_N \in L(V_N, \Phi_N), \quad \boldsymbol{f}_N = s_N f. \tag{3}$$

Operators $\boldsymbol{A}_N$ are called stable if there is independent of N constant $\mu > 0$ such that for arbitrary $\boldsymbol{u}_N \in V_N$ is performed inequality

$$\mu \|\boldsymbol{u}_N\|_{V_N} \leqslant \|\boldsymbol{A}_N \boldsymbol{u}_N\|_{F_N}. \tag{4}$$

Let us the pairs of operators (r_N, p_N) and (s_N, q_N) are selected. Assume as an approximate solution of equation (1) the function $p_N \boldsymbol{u}_N \in U_N$ where $\boldsymbol{u}_N$ is the solution of problem (3). Then we have the next basic theorem of convergence [29].

Theorem 1. *Let us the operator A is an isomorphism from U into F. Then the sequence $p_N \boldsymbol{u}_N$ converges to solution u of equation* (1) *if and only if the approximations (V_N, p_N, r_N) of the space U are convergent and operators A_N are stable. In addition, error estimation of the approximate solution is given by the ratio*

$$\|u - p_N \boldsymbol{u}_N\|_U \leqslant (1 + \|A\| / \mu) \|u - p_N r_N u\|_U.$$

The choice of triples (V_N, p_N, r_N) and (Φ_N, q_N, s_N) defines one or another projection method of solving the equation (1) (Galerkin, smallest squares, smallest mismatch, collocation, etc.).

3. B-splines approximations

Let us $S = [0, a] \times [0, b] \subset \mathbb{R}^2$. Construct in the domain S a rectangular grid S_h with the steps $h_1 = a/n$ and $h_2 = b/k$, $n, k = 1, 2, \ldots$. Introduce in the space $H^m(S)$, $m = 0, 1, 2, \ldots$, the system of B-splines of m-th degree

$$\{B_{ij}(\xi)\}_{i=-m}^{n-1}{}_{j=-m}^{k-1}, \quad n, k \geqslant m + 1. \tag{5}$$

Denote by U_B^N the linear shell of system (5). Select restriction operator $r_B^N \colon H^m(S) \to V_B^N \subset \mathbb{R}^N$ and extension operator $p_B^N \colon V_B^N \to U_B^N \subset H^m(S)$ in the form

$$(r_B^N v)_{i,j} \equiv v_N^{(i,j)} = \int_S B_{ij}(\xi) v(\xi)\, dS_\xi, \quad i = \overline{-m(1)(n-1)}, \quad j = \overline{-m(1)(k-1)}, \tag{6}$$

$$r_B^N v = v_B^N \in V_B^N, \quad v \in H^m(S),$$

$$p_B^N v_B^N = \sum_{i=-m}^{n-1} \sum_{j=-m}^{k-1} v_N^{(i,j)} B_{ij}(\xi), \quad N = (n+m)(k+m). \tag{7}$$

The next result is in order [29].

Lemma 2. *Approximations (V_B^N, p_B^N, r_B^N) of the space $H^m(S)$ are convergent and for arbitrary $v \in H^m(S)$ are performed the estimates*

$$\left\| v - p_B^N r_B^N v \right\|_{H^t(S)} \leqslant C h^{\sigma - t} \left\| v \right\|_{H^\sigma(S)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{8}$$

where constant $C > 0$ does not depend on v.

Denote by G a bounded open domain in $\mathbb{R}^3$ with boundary Γ. Suppose that exists M open balls

$$B_l \subset \mathbb{R}^3, \quad \Gamma \subset \bigcup_{l=1}^{M} B_l, \quad B_l \cap \Gamma = \Gamma_l \neq 0, \quad l = \overline{1, M},$$

such that for each ball B_l there is defined on $\bar{B}_l$ m times differentiated real vector-function $f^{(l)}(x) = (f_1^{(l)}, f_2^{(l)}, f_3^{(l)})$ such that $y_l = f^{(l)}(x)$ carries a mutually unambiguous mapping of the ball B_l onto some open bounded set in $\mathbb{R}^3$ where Γ_l is mapped on an open set $S_l \subset \mathbb{R}^2$. In addition, the Jacobian $J_l = \frac{\partial(f_1^{(l)}, f_2^{(l)}, f_3^{(l)})}{\partial(x_1, x_2, x_3)}$ is positive and continuous if $x \in \bar{B}_l$, $l = \overline{1, M}$. Then the surface Γ is called m-smooth surface in $\mathbb{R}^3$ [30].

We associate to the partition $\Gamma = \cup_{l=1}^{M} \Gamma_l$ the partition of one $\{\psi_l(x)\}_{l=1}^{M}$, $x \in \Gamma$, with the following properties:

$$\psi_l(x) \in C^\infty(\Gamma_l), \quad \operatorname{supp}\{\psi_l\} \subset \Gamma_l, \quad 0 \leqslant \psi_l(x) \leqslant 1, \quad l = \overline{1, M}, \quad \sum_{l=1}^{M} \psi_l(x) = 1,$$

and exists m times continuously differentiated mapping

$$\tau_l \colon \Gamma_l \to S_l = [0, a_l] \times [0, b_l], \quad \tau_l^{-1} \colon S_l \to \Gamma_l, \quad l = \overline{1, M}.$$

Then for arbitrary function $u(x)$ defined on Γ we can put into a mutually unambiguous correspondance the set of defined in $\mathbb{R}^2$ functions

$$\{v_l(\xi)\}_{l=1}^{M}, \quad v_l(\xi) = u(\tau_l^{-1}(\xi)), \quad \xi \in S_l,$$

which have the compact support on S_l and

$$u_l(x)=\psi_l(x)u(x),\quad x\in\Gamma,\quad u(x)\in H^m(\Gamma),\quad \text{if}\quad v_l(\xi)\in H^m(S_l),\quad l=\overline{1,M},$$

and

$$\|u\|_{H^m(\Gamma)}=\sum_{l=1}^{M}\|v_l\|_{H^m(S_l)}.$$

Construct in each domain S_l a rectangular grid S_l^h with the steps $h_1^{(l)}=a_l/n_l$ and $h_2^{(l)}=b_l/k_l$, and define in each grid domain S_l^h a system of functions

$$\{B_{ij}^{(l)}(\xi)\}_{i=-m}^{n_l-1}{}_{j=-m}^{k_l-1},\quad n_l,k_l\geqslant m+1, l=\overline{1,M}.$$

Assign to them the grid $\Gamma_h=\cup_{l=1}^{M}\tau_l^{-1}(S_l^h)$ on the surface Γ and system of functions

$$\{\tilde{B}_k(x)\}_{k=1}^{N_B}=\bigcup_{l=1}^{M}\{B_{ij}^{(l)}(\tau_l(x))\}_{i=-m}^{n_l-1}{}_{j=-m}^{k_l-1},\quad N_B=\sum_{l=1}^{M}(n_l+m)(k_l+m). \tag{9}$$

Denote $\Gamma_{l_1,l_2}=\Gamma_{l_1}\cap\Gamma_{l_2}$, $l_1,l_2=\overline{1,M}$, and suppose that the grid on surface Γ satisfies condition

$$\operatorname{supp}\{B_{ij}^{(l)}(\tau_l(x))\}\not\subset\Gamma_{l,k},\quad i=\overline{-m(1)(n_l-1)},\quad j=\overline{-m(1)(k_l-1)},\quad k\neq l,\quad k,l=\overline{1,M}.$$

Since $\operatorname{supp}\{\tilde{B}_{i_1}(x)\}\neq\operatorname{supp}\{\tilde{B}_{i_2}(x)\}$ for $i_1\neq i_2$, the functions of system $\{\tilde{B}_k(x)\}_{k=1}^{N_B}$ are linearly independent.

Denote by $\tilde{r}_B^{N_B}$ the restriction operator from $H^m(\Gamma)$ onto finite dimensional space $V_B^{N_B}$, and $\tilde{r}_B^{N_l}$ is its restriction to $H^m(\Gamma_l)$, that is,

$$\tilde{r}_B^{N_B}=\{\tilde{r}_B^{N_l}\}_{l=1}^{M},\quad \tilde{r}_B^{N_l}u_l(x)=r_B^{N_l}v_l(\xi),\quad N_l=(n_l+m)(k_l+m), \tag{10}$$

where $r_B^{N_l}$ is the similar to (6) restriction operator from $H^m(S_l)$ onto finite dimensional space $V_B^{N_l}$, $l=\overline{1,M}$, and $V_B^{N_B}=V_B^{N_1}\times V_B^{N_2}\times\ldots\times V_B^{N_M}$.

The extension operator $\tilde{p}_B^{N_B}$ from $V_B^{N_B}$ onto $U_B^{N_B}\subset H^m(\Gamma)$ is introduced by the formula

$$(\tilde{p}_B^{N_B}\boldsymbol{u}_{N_B})(x)=\sum_{i=1}^{N_B}u_i\tilde{B}_i(x),\quad \boldsymbol{u}_{N_B}\in V_B^{N_B}. \tag{11}$$

From lemma 2 follows that

$$\lim_{N_B\to\infty}\left\|u-\tilde{p}_B^{N_B}\tilde{r}_B^{N_B}u\right\|_{H^m(\Gamma)}=\sum_{l=1}^{M}\lim_{N_l\to\infty}\left\|v_l-p_B^{N_l}r_B^{N_l}v_l\right\|_{H^m(S_l)}=0,$$

i.e. approximations $(V_B^{N_B},\tilde{p}_B^{N_B},\tilde{r}_B^{N_B})$ of the space $H^m(\Gamma)$ are convergent. Further, from estimates (8) we obtain

$$\left\|u-\tilde{p}_B^{N_B}\tilde{r}_B^{N_B}u\right\|_{H^t(\Gamma)}^2=\sum_{l=1}^{M}\left\|v_l-p_B^{N_l}r_B^{N_l}v_l\right\|_{H^t(S_l)}^2\leqslant C^2h^{2(\sigma-t)}\sum_{l=1}^{M}\|v_l\|_{H^\sigma(S_l)}^2=C^2h^{2(\sigma-t)}\|u\|_{H^\sigma(\Gamma)}^2,$$
$$0\leqslant t\leqslant\sigma\leqslant m+1,\quad t\leqslant m,$$

where $p_B^{N_l}$ is the similar to (7) extension operator from $V_B^{N_l}$ onto the spaces $H^m(S_l)$, constant $C>0$ does not depend on u, and $h=\max_{1\leqslant l\leqslant M}\{h_1^{(l)}h_2^{(l)}\}$.

Thus, it is proved

Lemma 3. *Approximations* $(V_B^{N_B}, \tilde{p}_B^{N_B}, \tilde{r}_B^{N_B})$ *of the space* $H^m(\Gamma)$ *are convergent and for arbitrary* $u \in H^m(\Gamma)$ *are valid the estimates*

$$\left\| u - \tilde{p}_B^{N_B} \tilde{r}_B^{N_B} u \right\|_{H^t(\Gamma)} \leqslant C\, h^{\sigma - t} \left\| u \right\|_{H^\sigma(\Gamma)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{12}$$

in which constant $C > 0$ *does not depend on* u.

4. Lagrangian approximations

Assign to each element of the grid S_h of domain S

$$P_{ij} = [h_1 i, h_1(i+1)] \times [h_2 j, h_2(j+1)], \quad i = \overline{0(1)(n-1)}, \quad j = \overline{0(1)(k-1)},$$

a smaller rectangular grid P_{ij}^ε with the steps $\varepsilon_1 = h_1/m$ and $\varepsilon_2 = h_2/m$. Denote $S_{h,\varepsilon} = \cup_{i,j} P_{ij}^\varepsilon$ and associate with the set of nodes $S_{h,\varepsilon}$ a system of piecewise polynomial functions

$$\left\{ L_{pt}(\xi) \right\}_{p=0}^{mn}\,{}_{t=0}^{mk}, \tag{13}$$

satisfying conditions

$$L_{pt}(\xi_{ls}) = \delta_{pl}\delta_{ts}, \quad \operatorname{supp}\left\{ L_{pt}(\xi) \right\} = \tilde{P}_{pt}, \quad \tilde{P}_{pt} = \Big\{ \bigcup_{i,j} P_{ij} \colon \xi_{pt} \in P_{ij} \Big\}, \quad \xi_{ls} \in \tilde{P}_{pt}, \tag{14}$$

where δ_{pl} is the Kronecker symbol.

Functions (13)–(14) form a system of Lagrangian finite elements of m-th degree in $H^m(S)$. Denote by $U_L^{N_1}$ the linear shell of this system, $N_1 = (1 + mn)(1 + mk)$. It is obvious that the restriction of system (13)–(14) onto an arbitrary rectangle P_{ij} of the grid S_h is a basis in the space of polynomials $\mathrm{P}^m(P_{ij})$ of degree not higher than m, defined on P_{ij}. Then

$$U_B^N \subset U_L^{N_1}. \tag{15}$$

Choose the extension operator $p_L^{N_1} \colon V_L^{N_1} \to U_L^{N_1} \subset H^m(S)$, where $V_L^{N_1} \subset \mathbb{R}^{N_1}$, in the form

$$p_L^{N_1} v_L^{N_1} = \sum_{i=0}^{mn} \sum_{j=0}^{mk} v_{N_1}^{(i,j)} L_{ij}(\xi), \quad v_L^{N_1} = \left(v_{N_1}^{(0,0)}, v_{N_1}^{(0,1)}, \ldots, v_{N_1}^{(mn,mk)} \right). \tag{16}$$

Then, by virtue of the embedding (15), there exists a restriction operator $r_L^{N_1} \colon H^m(S) \to V_L^{N_1}$ such that approximations $(V_L^{N_1}, p_L^{N_1}, r_L^{N_1})$ of the space $H^m(S)$ are convergent and valid the estimates

$$\left\| v - p_L^{N_1} r_L^{N_1} v \right\|_{H^t(S)} \leqslant \tilde{C} h^{\sigma - t} \left\| v \right\|_{H^\sigma(S)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{17}$$

in which constant $\tilde{C} > 0$ does not depend on v.

Thus, it is proved

Lemma 4. *There is a restriction operator* $r_L^{N_1} \colon H^m(S) \to V_L^{N_1}$ *such that approximations* $(V_L^{N_1}, p_L^{N_1}, r_L^{N_1})$ *of the space* $H^m(S)$ *are convergent and valid the estimates* (17).

Assume that surface Γ satisfy the conditions of p. 3. Construct in each domain S_l the rectangular grid S_l^h with the steps $h_1^{(l)} = a_l/n_l$ and $h_2^{(l)} = b_l/k_l$ and set on each element P_{ij}^l of the grid S_l^h a smaller grid with the steps $\varepsilon_1^{(l)} = h_1^{(l)}/m$ and $\varepsilon_2^{(l)} = h_2^{(l)}/m$, $l = \overline{1, M}$. Define analogously to (13), (14) in each grid domain $S_l^{h,\varepsilon} = \cup_{i,j} P_{ij}^{l,\varepsilon}$ the system of Lagrangian finite elements

$$\{L_{ij}^{(l)}(\xi^{(l)})\}_{i=0}^{n_l m}\,{}_{j=0}^{k_l m}, \quad \xi^{(l)} \in S_l, \quad l = \overline{1,M}.$$

Assign to the family $S_l^{h,\varepsilon}$ the grid $\Gamma_{h,\varepsilon} = \cup_{l=1}^M \tau_l^{-1}(S_l^{h,\varepsilon})$ on the surface Γ, where $\tau_l^{-1}(P_{ij}^{l,\varepsilon})$ are the elements of the grid $\Gamma_{h,\varepsilon}$, $l = \overline{1,M}$. Denote by T_l the set of nodes of the grid $S_l^{h,\varepsilon}$, $l = \overline{1,M}$, $T = \cup_{l=1}^M T_l$. We number all elements of the set T with the cross-cutting index $t = \overline{1,K}$, $K = \sum_{l=1}^M K_l$, $K_l = (1+n_l m)(1+k_l m)$, and put in correspondence to each node x_p of the grid $\Gamma_{h,\varepsilon}$ the set of elements

$$P_p^* = \Big\{P_{ij}^{l,\varepsilon} \subset \bigcup_{l=1}^M S_l^{h,\varepsilon} \colon x_p \in \tau_l^{-1}(P_{ij}^{l,\varepsilon})\Big\},$$

element

$$\tilde{P}_p = \Big\{\bigcup_{i,j} \tau_l^{-1}(P_{ij}^{l,\varepsilon}), P_{ij}^{l,\varepsilon} \in P_p^*, l = \overline{1,M}\Big\},$$

the set of indexes

$$T_p^* = \Big\{t \in T \colon \tau_l^{-1}(P_{ij}^{l,\varepsilon}) = x_p,\ \xi_t^{(l)} \in P_l^{h,\varepsilon}\ l = \overline{1,M}\Big\},$$

and function

$$\tilde{L}_p(x) = \sum_{t \in T_p^*} L_t^l(\tau_l(x)), \quad x \in \Gamma_l, \quad \operatorname{supp}\{\tilde{L}_p(x)\} = \tilde{P}_p, \quad L_t^{(l)}(\xi^{(l)}) \in \{L_{ij}^{(l)}(\xi^{(l)})\}_{i=0}^{n_l m}\,{}_{j=0}^{k_l m}.$$

Denote by $\tilde{r}_L^{N_L}$ the restriction operator from $H^m(\Gamma)$ into the finite dimensional space $V_L^{N_L}$ and by $\tilde{r}_L^{N_l}$ — its restriction to $H^m(\Gamma_l)$, i.e.

$$\tilde{r}_L^{N_L} = \{\tilde{r}_L^{N_l}\}_{l=1}^M, \quad \tilde{r}_L^{N_l} u_l(x) = r_L^{K_l} v_l(\xi), \tag{18}$$

where $r_L^{K_l}$ is the restriction operator from $H^m(S_l)$ into the corresponding finite dimensional space $V_L^{K_l}$, $l = \overline{1,M}$, and N_L is the number of nodes in the grid $\Gamma_{h,\varepsilon}$.

The extraction operator $\tilde{p}_L^{N_L}$ from $V_L^{N_L}$ into the linear shell $U_L^{N_L}$ of the system $\{\tilde{L}_p(x)\}_{p=1}^{N_L}$, $U_L^{N_L} \subset H^m(\Gamma)$, introduce by formula

$$\big(\tilde{p}_L^{N_L} u_L^{N_L}\big)(x) = \sum_{i=1}^{N_L} u_N^{(i)} \tilde{L}_i(x), \quad u_L^{N_L} \in V_L^{N_L}. \tag{19}$$

From Lemma 4 follows that

$$\lim_{N_L \to \infty} \Big\|u - \tilde{p}_L^{N_L} \tilde{r}_L^{N_L} u\Big\|_{H^m(\Gamma)} = \sum_{l=1}^M \lim_{N_l \to \infty} \Big\|v_l - p_L^{N_l} r_L^{N_l} v_l\Big\|_{H^m(S_l)} = 0,$$

i.e. approximations $(V_L^{N_L}, \tilde{p}_L^{N_L}, \tilde{r}_L^{N_L})$ of the space $H^m(\Gamma)$ are convergent. Further from estimate (17) we obtain

$$\Big\|u - \tilde{p}_L^{N_L} \tilde{r}_L^{N_L} u\Big\|_{H^t(\Gamma)}^2 = \sum_{l=1}^M \Big\|v_l - p_L^{N_l} r_L^{N_l} v_l\Big\|_{H^t(S_l)}^2 \leqslant \tilde{C}^2 h^{2(\sigma-t)} \sum_{l=1}^M \|v_l\|_{H^\sigma(S_l)}^2 = \tilde{C}^2 h^{2(\sigma-t)} \|u\|_{H^\sigma(\Gamma)}^2, \quad 0 \leqslant t \leqslant \sigma \leqslant m,$$

where $p_L^{N_l}$ is a similar to (16) extension operator from $V_L^{N_l}$ into $H^m(S_l)$, constant $\tilde{C} > 0$ does not depend on u and $h = \max\limits_{1 \leqslant l \leqslant M} \{h_1^{(l)} h_2^{(l)}\}$.

Thus, it is proved

Lemma 5. *There is a restriction operator* $\tilde r_L^{N_L}\colon H^m(\Gamma) \to V_L^{N_L}$ *such that approximations* $(V_L^{N_L}, \tilde p_L^{N_L}, \tilde r_L^{N_L})$ *of the space* $H^m(\Gamma)$ *are convergent and valid the estimates*

$$\left\| u - \tilde p_L^{N_L} \tilde r_L^{N_L} u \right\|_{H^t(\Gamma)} \leqslant \tilde C h^{\sigma - t} \left\| u \right\|_{H^\sigma(\Gamma)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{20}$$

in which constant $\tilde C > 0$ *does not depend on* u.

5. Galerkin method

Let us denote $G' = \mathbb{R}^3 \backslash \bar G$ and introduce in G and G' the Sobolev spaces [30]

$$H^m(G) = \{v \in L_2(G)\colon \partial^\alpha v \in L_2(G),\ |\alpha| \leqslant m\},$$

$$W^m(G') = \{v \in D'(G')\colon (1 + r^2)^{(|\alpha|-1)^2} \partial^\alpha v \in L_2(G'),\ |\alpha| \leqslant m\},$$

where $m \geqslant 0$, and $r = \left(\sum_{i=1}^3 x_i^2\right)^{1/2}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Consider the next boundary value problem: to find function

$$v \in H^{m+1}_{\Gamma,\Delta=0} = \left\{ v \in H^{m+1}(G) \bigcup W^{m+1}(G') : v|_{\Gamma_{int}} = v|_{\Gamma_{ext}},\ \Delta v(x) = 0,\ x \in G, G' \right\} \tag{21}$$

satisfying condition

$$v|_\Gamma = f, \quad f \in H^{m+1/2}(\Gamma). \tag{22}$$

In [9] was proved the next

Theorem 6. *Problem* (21)–(22) *has one and only one solution. We will search a solution of the problem* (21)–(22) *in the form of simple layer potential*

$$v(x) = \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} d\Gamma_y, \quad x \in G, G'.$$

The unknown potential density is determined from the equation

$$(\mathrm{A}u)(x) \equiv \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} d\Gamma_y = f(x), \quad x \in \Gamma. \tag{23}$$

The next result is in order [9].

Theorem 7. *Operator* A *is an isomorphism of* $H^s(\Gamma)$ *onto* $H^{s+1}(\Gamma)$, $s \geqslant -1/2$. *From the last statement and the Banach theorem follows the validity of inequalities*

$$\alpha_s \left\| u \right\|_{H^s(\Gamma)} \leqslant \left\| Au \right\|_{H^{s+1}(\Gamma)} \leqslant \beta_s \left\| u \right\|_{H^s(\Gamma)}, \tag{24}$$

in which constants α_s *and* β_s, $0 < \alpha_s \leqslant \beta_s$, *does not depend on* $u \in H^s(\Gamma)$.

Suppose that for approximation of unknown potential density $u \in H^m(\Gamma)$ uses the system of B-splines of the form (9), and U_{N_B} is its linear shell. We choose the operators $\tilde r_{N_B}\colon H^m(\Gamma) \to V_{N_B}$ and $\tilde p_{N_B} : V_{N_B} \to U_{N_B}$ in the form (10) and (11) respectively and determine the restriction operator $s_{N_B}\colon H^{m+1}(\Gamma) \to \Phi_{N_B}$ in the form $s_{N_B} = \tilde r_{N_B}$. In this case, the system

$$\boldsymbol{A}^G_{N_B} \boldsymbol{u}_{N_B} = \boldsymbol{f}_{N_B}, \quad \boldsymbol{A}^G_{N_B} = \tilde r_{N_B} A \tilde p_{N_B}, \quad \boldsymbol{f}_{N_B} = r_{N_B} f,$$

implements Galerkin method of solving the equation (23). From Lax-Milgram lemma [31] follows that matrix $\boldsymbol{A}^G_{N_B}$ is nondegenerate and, accordingly, the definition of operator q_{N_B} in the form $q_{N_B} \boldsymbol{f}_{N_B} =$

$A\tilde{p}_{N_B}\boldsymbol{u}_{N_B}$ is correct. Taking into account the left side of inequalities (24), the bijectivity of mapping $\tilde{p}_{N_B} : V_{N_B} \to U_{N_B}$, the expressions for the norms in the spaces V_{N_B} and Φ_{N_B} in the case $U = H^m(\Gamma)$, $F = H^{m+1}(\Gamma)$, and equality $Q_{N_B}AP_{N_B}u = AP_{N_B}u$, we obtain the following inequalities

$$\alpha_m \left\|\boldsymbol{u}_{N_B}\right\|_{V_{N_B}} \leqslant \left\|\boldsymbol{A}^G_{N_B}\boldsymbol{u}_{N_B}\right\|_{\Phi_{N_B}} \tag{25}$$

for arbitrary $\boldsymbol{u}_{N_B} \in V_{N_B}$ and α_m does not depend on $\boldsymbol{u}_{N_B}$.

Then from the inequalities (24) and (25), Lemma 3 and Theorem 1 we obtain the validity of following statement.

Theorem 8. *For arbitrary $f \in H^{m+1}(\Gamma)$, $m = 0, 1, \ldots,$ the approximate solution $u^B_{N_B}$ of equation* (23) *obtained by the Galerkin method under approximation of unknown potential density by the system of functions constructed on the basis of B-splines of m-th degree converges to its exact solution, and there is an estimate*

$$\left\|u - u^B_{N_B}\right\|_{H^t(\Gamma)} \leqslant \frac{C(1 + \beta_t/\alpha_t)}{\alpha_\sigma} h^{\sigma-t} \left\|f\right\|_{H^{\sigma+1}(\Gamma)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{26}$$

where h is the maximum area of the grid element on Γ.

Similarly, from the inequalities (24) and (25), Lemma 5 and Theorem 1, we obtain the validity of following statement.

Theorem 9. *For arbitrary $f \in H^{m+1}(\Gamma)$, $m = 0, 1, \ldots,$ the approximate solution $u^L_{N_L}$ of equation* (23) *obtained by the Galerkin method under approximation of unknown potential density by the system of functions constructed on the basis of Lagrangian finite elements of m-th degree converges to its exact solution, and there is an estimate*

$$\left\|u - u^L_{N_L}\right\|_{H^t(\Gamma)} \leqslant \frac{\tilde{C}(1 + \beta_t/\alpha_t)}{\alpha_\sigma} h^{\sigma-t} \left\|f\right\|_{H^{\sigma+1}(\Gamma)}, \quad 0 \leqslant t \leqslant \sigma \leqslant m, \tag{27}$$

where h is the maximum area of the grid element on Γ.

6. Collocation method

To simplify the presentation, we assume that for approximation of unknown potential density $u \in H^m(\Gamma)$, $m \geqslant 0$, of equation (23) a system of linearly independent functions $\{\varphi_i\}_{i=1}^{\infty}$ is chosen, U_N is a linear shell of the system $\{\varphi_i\}_{i=1}^{N}$, $r_N \colon H^m(\Gamma) \to V_N$, $p_N \colon V_N \to U_N$ are the similar to described in p. 2 restriction and extraction operators. Denote by X_N the set of pairwise different points belonging to the surface Γ

$$X_N = \{x_j\}_{j=1}^N, \quad x_j \in \Gamma, \quad j = \overline{1, N},$$

and introduce in $H^{m+1}(\Gamma)$ restriction operator $s_N \colon H^{m+1}(\Gamma) \to \Phi_N$ by formula

$$(s_N f)_j = f(\tilde{y}_j) \tag{28}$$

in which

$$\tilde{y}_j \in \left\{\tilde{y} \in \delta(y_j) \colon \left|f(\tilde{y})\right| = \min_{y \in \delta(y_j)} \left|f(y)\right|, \, y_j \in X_N\right\}, \quad \delta(y_j) = \{y \in \Gamma \colon \left|y - y_j\right| < \delta\}, \tag{29}$$

in particular $\rho(y^*, \delta(y_j)) > 0$ for arbitrary $y^* \in X_N$, $y^* \neq y_j$, $j = \overline{1, N}$.

If $f \in C(\Gamma)$, then operator s_N can be defined as usual

$$(s_N f)_j = f(y_j), \quad y_j \in X_N, \tag{30}$$

i.e. $\tilde{y}_j = y_j$, $j = \overline{1,N}$. It is easy to see that, with this choice of operator s_N, a system of linear algebraic equations

$$A_N^c u_N = s_N f, \quad A_N^c = s_N A p_N, \quad u_N \in V_N, \tag{31}$$

implements the collocation method of solving the equation (23). The set X_N is called a set of collocation points.

Denote $Y_N = \{\tilde{y}_j\}_{j=1}^N$ and consider the system of functions

$$r_j(x) = \frac{1}{|x - \tilde{y}_j|}, \quad \tilde{y}_j \in Y_N, \quad j = \overline{1,N}.$$

From the choice of the set X_N and conditions (29) follow that the functions of system $\{r_j(x)\}_{j=1}^N$ are linearly independent [32].

Define in $L^\infty(\Gamma)$ the family of linear continuous functionals

$$l_j(\varphi) = \int_\Gamma \varphi(x) r_j(x) d\Gamma_x, \quad \varphi \in L^\infty(\Gamma), \quad j = \overline{1,N}.$$

Denote by $\operatorname{Ker}(l_j)$ the zero subspace of functional l_j in $L^\infty(\Gamma)$

$$\operatorname{Ker}(l_j) = \{\varphi \in L^\infty(\Gamma) \colon l_j(\varphi) = 0\}$$

and suppose that $K_N = \cap_{j=1}^N \operatorname{Ker}(l_j)$. The degeneracy of matrix A_N^c is equivalent to the linear dependence of its rows or columns, that is, the existence of such sets $\alpha_N = \{\alpha_i\}_{i=1}^N \in \mathbb{R}^N$ or $\beta_N = \{\beta_j\}_{j=1}^N \in \mathbb{R}^N$, $\sum_{i=1}^N \alpha_i^2 > 0$, $\sum_{j=1}^N \beta_j^2 > 0$, that

$$\int_\Gamma \Big(\sum_{i=1}^N \alpha_i \varphi_i(x) \Big) r_j(x) d\Gamma_x = 0, \quad j = \overline{1,N}, \tag{32}$$

or

$$\int_\Gamma \varphi_i(x) \Big(\sum_{j=1}^N \beta_i r_j(x) \Big) d\Gamma_x = 0, \quad i = \overline{1,N}. \tag{33}$$

Implementation of equations (32), (33) is only possible if $K_N \cap U_N \neq 0$. From this follows sufficient conditions for the invertibility of matrix A_N^c, which we formulate in the next statement.

Lemma 10. *Let us the system of linearly independent functions $\{\varphi_i\}_{i=1}^N$ is chosen for the approximate solution of equality (23) and determined the set of collocation points X_N (and, consequently, the set K_N is defined). Then, if*

$$K_N \cap U_N = 0, \tag{34}$$

then the matrix A_N^c of the system of collocation equations (31) is non-degenerate for arbitrary N.

A similar result is obtained if the restriction operator s_N is chosen in the form

$$(s_N f)_j = \frac{1}{\operatorname{mes} \delta(y_j)} \int_{\delta(y_j)} f(y) d\Gamma_y \tag{35}$$

and

$$r_j(x) = \frac{1}{\operatorname{mes} \delta(y_j)} \int_{\delta(y_j)} \frac{d\Gamma_y}{|x - y|}, \quad j = \overline{1,N}.$$

It is obvious that under conditions of Lemma 10 the operator A_N^c, where s_N is defined according to (27)–(28) or (33), or in the case of $f \in C(\Gamma)$ according to (30), is stable in sense (4).

Consider a discrete analog of condition (34). Let us the quadrature formula

$$\int_\Gamma \varphi(x) r_i(x) d\Gamma_x \approx \sum_{j=1}^{N} A_j \varphi(x_j) r_i(x_j), \quad x_j \in \Gamma, \quad x_j \neq x_i, \quad \text{if} \quad j \neq i, \tag{36}$$

is used to calculate the integrals

$$\int_\Gamma \varphi(x) r_i(x) d\Gamma_x, \quad \varphi(x) \in U_N, \quad i = \overline{1,N},$$

which is exact for integrals

$$\int_\Gamma \varphi(x) \psi(x) d\Gamma_x, \quad \varphi(x), \psi(x) \in U_N.$$

Consider the system of functions

$$\psi_i(x) = \sum_{k=1}^{N} \alpha_k^{(i)} \varphi_k(x), \tag{37}$$

the coefficients $\alpha_k^{(i)}$, $k, i = \overline{1,N}$, of which we define from N systems of linear algebraic equations

$$\sum_{k=1}^{N} \alpha_k^{(i)} \varphi_k(x_j) = r_i(x_j), \quad i, j = \overline{1,N}. \tag{38}$$

Define the conditions under which the functions $\psi_i(x)$, $i = \overline{1,N}$, are linearly independent. From (37) we obtain that

$$\sum_{i=1}^{N} c_i \psi_i(x) = \sum_{k=1}^{N} \Big(\sum_{i=1}^{N} c_i \alpha_k^{(i)} \Big) \varphi_k(x) = 0$$

if and only if

$$\sum_{i=1}^{N} c_i \alpha_k^{(i)} = 0, \quad k = \overline{1,N}. \tag{39}$$

Let us the set of colocation points $X_N = \{y_j\}_{j=1}^N \subset \Gamma$ is chosen in such a way that

$$0 < |x_i - y_i| < \varepsilon, \quad d < |x_i - y_j|, \quad i, j = \overline{1,N}, \quad j \neq i, \quad 0 < \varepsilon < \frac{d}{N-1},$$

where $\{x_j\}_{j=1}^N$ are the nodes of quadrature formula (36). Then

$$r_i(x_i) > \sum_{i=1, i \neq j}^{N} r_i(x_j),$$

matrix $\boldsymbol{R}_N = \{r_i(x_j)\}_{i,j=1}^N$ due to Hadamard condition is nondegenerate and from (38) we obtain that vectors $\boldsymbol{\alpha}_k = \{\alpha_k^{(j)}\}_{j=1}^N$, $k = \overline{1,N}$, are linearly independent. Hence, equality (39) holds if and only if $c_i = 0$, $i = \overline{1,N}$, i.e. the functions of system $\{\psi_i(x)\}_{i=1}^N$ are linearly independent.

Now, if the quadrature formula of form (36) is used to calculate the integrals in coefficients of matrix $\boldsymbol{A}_N^c$, instead of the system of collocation equations (31), we actually solve a system with matrix

$$\tilde{\boldsymbol{A}}_N^c = \Big\{ \int_\Gamma \varphi_i(x) \psi_j(x) d\Gamma_x \Big\}_{i,j=1}^N,$$

where functions $\psi_i(x)$, $i=\overline{1,N}$, are defined by formulas (37) and (38). The last matrix can be degenerate if and only if there exists a nonzero element $\varphi(x)=\sum_{i=1}^{N} a_i\varphi_i(x)\in U_N$, orthogonal to all $\psi_i(x)$, $i=\overline{1,N}$, which is impossible, since the system $\{\psi_i(x)\}_{i=1}^{N}$ forms a basis in the space U_N.

Let us the system of B-splines of the form (9) is used to approximate the unknown potential density $u\in H^m(\Gamma)$ and U_{N_B} is its linear shell. We choose the operators $\tilde r_{N_B}\colon H^m(\Gamma)\to V_{N_B}$ and $\tilde p_{N_B}: V_{N_B}\to U_{N_B}$ in the form (18) and (19) respectively and determine the restriction operator $s_{N_B}\colon H^{m+1}(\Gamma)\to \Phi_{N_B}$ in the form (28), (29). In this case, the system

$$\boldsymbol{A}^c_{N_B}\boldsymbol{u}_{N_B}=\boldsymbol{f}_{N_B},\quad \boldsymbol{A}^c_{N_B}=\tilde r_{N_B}A\tilde p_{N_B},\quad \boldsymbol{f}_{N_B}=r_{N_B}f,$$

implements the collocation method for solution of equation (23). From Lax-Milgram lemma [31] follows that under conditions (34) matrix $\boldsymbol{A}^c_{N_B}$ is non-degenerate and, accordingly, the definition of operator q_{N_B} in the form $q_{N_B}\boldsymbol{f}_{N_B}=A\tilde p_{N_B}\boldsymbol{u}_{N_B}$ is correct. Given the left side of inequalities (24), the biectivity of mapping $\tilde p_{N_B}\colon V_{N_B}\to U_{N_B}$, the expressions for norms in the spaces V_{N_B} and Φ_{N_B} in the case $U=H^m(\Gamma)$, $F=H^{m+1}(\Gamma)$ and equality $Q_{N_B}AP_{N_B}u=AP_{N_B}u$, we obtain the validity of inequalities (25) for arbitrary $\boldsymbol{u}_{N_B}\in V_{N_B}$, in which α_m does not depend on $\boldsymbol{u}_{N_B}$.

Then from the inequalities (24) and (25), Lemmas 3, 10, and Theorem 1 we obtain the validity of following statement.

Theorem 11. *For arbitrary $f\in H^{m+1}(\Gamma)$, $m=0,1,\dots$, the approximate solution $u^B_{N_B}$ of equation* (23) *obtained by collocation method under approximation of unknown potential density by a system of functions constructed on the basis of B-splines of m-th degree and the choice of collocation points that satisfies the condition* (34) *converges to its exact solution, and there is an estimate*

$$\left\|u-u^B_{N_B}\right\|_{H^t(\Gamma)}\leqslant \frac{C(1+\beta_t/\alpha_t)}{\alpha_\sigma}h^{\sigma-t}\left\|f\right\|_{H^{\sigma+1}(\Gamma)},\quad 0\leqslant t\leqslant\sigma\leqslant m, \tag{40}$$

where h is the maximum area of the grid element on Γ.

Similarly, from the inequalities (24) and (25), Lemmas 5, 10, and Theorem 1 we obtain the validity of following statement.

Theorem 12. *For arbitrary $f\in H^{m+1}(\Gamma)$, $m=0,1,\dots$, the approximate solution $u^L_{N_L}$ of equation* (23) *obtained by collocation method under approximation of unknown potential density by a system of functions constructed on the basis of Lagrangian finite elements of m-th degree and the choice of collocation points that satisfies the condition* (34) *converges to its exact solution, and there is an estimate*

$$\left\|u-u^L_{N_L}\right\|_{H^t(\Gamma)}\leqslant \frac{\tilde C(1+\beta_t/\alpha_t)}{\alpha_\sigma}h^{\sigma-t}\left\|f\right\|_{H^{\sigma+1}(\Gamma)},\quad 0\leqslant t\leqslant\sigma\leqslant m, \tag{41}$$

where h is the maximum area of the grid element on Γ.

7. Error estimation of approximate solution of the Dirichlet problem for the Laplace equation

Denote by $u_N(x)$ the approximate solution of equation (23), obtained by means of considered above Galerkin or collocation methods, $N=N_B$ in the case of approximation by B-splines and $N=N_L$ in the case of Lagrangian approximations. Denote

$$v_N(x)=\frac{1}{4\pi}\int_\Gamma \frac{u_N(y)}{|x-y|}d\Gamma_y,\quad x\in G,G',$$

and estimate the modulus of value

$$\frac{\partial^\alpha}{\partial x^\alpha}(v(x)-v_N(x))=\frac{1}{4\pi}\int_\Gamma (u(y)-u_N(y))\frac{\partial^\alpha}{\partial x^\alpha}\frac{1}{|x-y|}d\Gamma_y, \quad x\in G,G', \quad \alpha=0,1,\dots.$$

Let us

$$x\in\mathbb{R}^3\backslash\{\tilde{x}\in\mathbb{R}^3\colon\ |\tilde{x}-y|<\delta, \quad y\in\Gamma\}. \tag{42}$$

Using Holder inequality, we obtain

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}(v(x)-v_N(x))\right|\leqslant\|u-u_N\|_{L_2(\Gamma)}\sqrt{\int_\Gamma\frac{\partial^\alpha}{\partial x^\alpha}\frac{1}{|x-y|}d\Gamma_y}, \quad x\in G,G',$$

or, taking into account (42),

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}(v(x)-v_N(x))\right|\leqslant\frac{\operatorname{mes}\Gamma}{\delta^{\alpha+1}}\|u-u_N\|_{L_2(\Gamma)}, \quad x\in G,G', \quad \alpha=0,1,\dots. \tag{43}$$

Then from inequalities (24), (43) and Theorems 8–12 follow the validity of the next statement.

Theorem 13. *For arbitrary $f\in H^{m+1}(\Gamma)$, $m=0,1,\dots$, an approximate solution of the problem* (21), (22) *obtained by Galerkin or collocation methods under approximation of unknown potential density by systems of functions constructed on the basis of B-splines or Lagrangian finite elements of the m-th degree, converges to its exact solution, and there is an estimate*

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}(v(x)-v_N(x))\right|\leqslant\frac{C^*(1+\beta_0/\alpha_0)h^m}{\alpha_m\delta^{\alpha+1}}\|f\|_{H^{m+1}(\Gamma)}, \quad x\in G,G', \quad \alpha=0,1,\dots.$$

8. Conclusions

The paper describes the conditions and evaluations of convergence of Galerkin and collocation methods for solution of Fredholm integral equation of the first kind for the simple layer potential in case of closed boundary surface in a three-dimensional space. Approximation of potential density was performed using B-splines and Lagrangian finite elements of various orders on rectangular grids constructed in the desired function definition domain. Estimations were obtained for the error of approximate solution of Dirichlet problem for Laplace equation that is equivalent to the integral equation for the simple layer potential. The approach proposed can be used to define convergence of other projection methods (the smallest squares, smallest mismatch etc.) for solving potential theory integral equations that are equivalent to the boundary value problems for equations of mathematical physics and other types of finite elements of various orders, constructed on both rectangular and triangular grids in desired potential density definition domain.

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Скінченно-елементні апроксимації у проекційних методах розв'язання деяких інтегральних рівнянь Фредгольма першого роду

Поліщук О.

Інститут прикладних проблем механіки і математики ім. Я. С. Підстригача, Національна академія наук України, вул. Наукова, 3-б, 79060, Львів, Україна

Досліджено апроксимаційні властивості B-сплайнів та лагранжевих кінцевих елементів у гільбертових просторах функцій, визначених на поверхнях у тривимірному просторі. Встановлено умови збіжності методів Гальоркіна та колокації розв'язання інтегрального рівняння Фредгольма першого роду для потенціалу простого шару, еквівалентного задачі Діріхле для рівняння Лапласа в $\mathbb{R}^3$. Визначено оцінку похибки наближеного розв'язку цієї задачі, отриманого за допомогою методів теорії потенціалу.

Ключові слова: *потенціал, інтегральне рівняння, коректна розв'язність, лагранжева апроксимація, B-сплайн, метод Гальоркіна, метод колокації, збіжність.*

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