COMPUTATIONAL PROBLEMS OF ELECTRICAL ENGINEERING

Vol. 7, No. 2, 2017

FOURIER SERIES OF PERIODIC FUNCTIONS WITH VARIABLE PERIOD AND EVALUATION OF THE VARIABLE PERIOD FOR DETERMINATION OF HEART RHYTHM VARIABILITY

Mykola Pryimak[1], Yaroslav Vasylenko[2], Lesia Dmytrotsa[1], Mariya Oliynyk[1]

[1]Ivan Puliuy National Technical University, Ternopil, Ukraine

[2]Volodymyr Hnatiuk National Pedagogical University, Ternopil, Ukraine

yava07@gmail.com, dmytrotsa.lesya@gmail.com

Abstract: The article draws attention to the fact that in addition to periodic empirical signals, whose model is a periodic function, there are signals that behave like periodic, but the period of their values repetition is no longer constant and changes in some way. An illustrative example is electrocardiograms (ECGs) obtained during or after an impact of some “exciter of calm”, for example, physical exertion, on the patient. How to study periodic signals with a variable period (PSVP)? The literature review shows that until recently there has been no scientifically substantiated answer to this question. Therefore, the problem of developing information technologies (IT) for doing research into PSVP is relevant both from theoretical and applied point of view. To solve the problem, we propose to use an approach, whose essence is the triad “model-algorithm-program”. Certain results in this direction have already been achieved in our previous works. Particularly, we give a definition of periodic functions with a variable period (PFVP), consider examples of trigonometric FVP (TFVP) and record their variable periods, develop a method for the formation of orthogonal TFVP system, and determine a scalar product for the functions of the system. In this paper, a Fourier series for PFVP is written, and formulas for finding its coefficients are obtained. As an example, a finite Fourier series is constructed for the analytically given PFVP, and it is shown that with number of coefficients increasing, the series approaches the function itself, which confirms the correctness of the theoretical results obtained.

Taking into account that for the vast majority of empirical PFVP their variable period is unknown, the question of its evaluation is raised. For the case of an ECG, obtained after physical activity, evaluations of its variable frequency (VF) and variable period (VP) are derived. The evaluation of a VF turned out to have the form of exponential function, which is determined by three parameters. The IT developed for the study of PFVP provide the opportunity to explore real PSVP, in particular, ECGs with VP, and the obtained numerical values of the parameters can be used in diagnostic tasks and decision making support.

Key words: variable period, electrocardiogram with a variable period, periodic functions with a variable period, system of trigonometric functions with a variable period, Fourier series of functions with a variable period.

1. Introduction

In the study of periodic functions, their period is traditionally considered to be constant. It is for this case that the theory of periodic functions, in particular, the theory of Fourier series has been developed, and the corresponding methods and algorithms for the analysis of such functions have been devised. At the same time, applied researches have to deal with empirical signals characterized by two features – they behave like periodic, i.e. there is a repeatability of their values, but the period of this repeatability changes in some way. An illustrative example of such signals is well-known ECGs obtained during or after physical exertion (or other “exciter of calm”) exposed to the patient. Under physical exertion, the heart rate increases to a certain value, i.e. the period decreases. After physical exertion, the heart rate, on the contrary, slows down, i.e. the period increases, approaching its “norm”. How to investigate the signals with a variable period? The literature review shows that at present only first but rather significant steps have been taken in this direction [1–5].

Referring to the above-mentioned works [1–5], as well as observing the basics of the theory of constructive functions [6, 7], one can assert that the most suitable approach to study the PSVP is the one, whose essence consists in the triad “model-algorithm-program”. At the first step, a model of the signal is substantiated, at the second one, an algorithm of the research into the model is developed, and the third stage is intended for creating algorithm implementation software. The first two components of the triad – model substantiation and algorithm development – are main here.

The purpose of the work is to develop an information technology for constructing a Fourier series for the periodic functions with a variable period.

2. Periodic functions with a variable period

Let us remember some of the results associated with the study of PSVP. First of all, this is a model for PSVP in the form of PFVP [1].

Definition. A function $f(x)$, $x \in I$ is called periodic with a variable period if there exists such a function $T(x) > 0$ that, for arbitrary points x and $x + T(x)$ from the domain of definition I, the values of function $f(x)$ at these points are equal, i.e. $f(x) = f(x + T(x))$. The function $T(x)$ is called a variable period.

The variable period $T(x)$ must satisfy certain conditions [3–5], in particular, be continuously differentiated, and its derivative must be $T'(x) > -1$.

An example of the variable period $T(x)$ is shown in Fig. 1. At point x_1, the period of the function $f(x)$ is equal to $T(x_1)$, so the value of the function at points x_1 and $x_1 + T(x_1)$ are equal: $f(x_1) = f(x_1 + T(x_1))$. At point x_2, the period is equal to $T(x_2)$, with the values of the period $T(x)$ at points x_1 and x_2 being different: $T(x_1) > T(x_2)$.

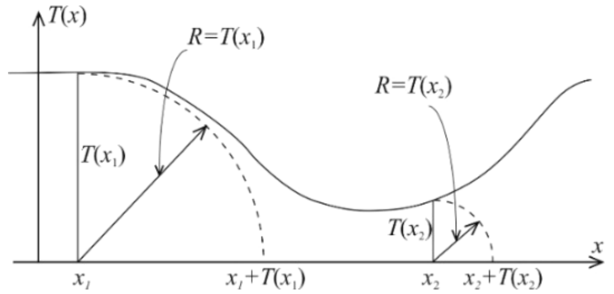

Fig. 1. The variable period $T(x)$, its values at points x_1 and x_2, and their corresponding points $x_1 + T(x_1)$ at which the values of the function $f(x)$ are repeated.

It is known that for a periodic function $g(x)$ with a constant period T, an equality $g(x) = g(x+T) = g(x-T)$ is satisfied For the function $f(x)$ with the variable period $T(x)$, the analogical equality $f(x) = f(x+T(x)) = f(x - T(x))$, in general, is not satisfied. Therefore, for the case when the argument x decreases, the variable period of repeatability of the function $f(x)$ is denoted $T^{-}(x)$. Herewith, if $x - T^{-}(x) > 0$, i.e. the argument belongs to the domain of definition I, then $f(x) = f(x - T^{-}(x))$. Between $T(x)$ and $T^{-}(x)$ there is a relationship [4,5], which is expressed by the following formulas: $T(x) = T^{-}(x + T(x))$, $T^{-}(x) = T(x - T^{-}(x))$.

3. Examples of analytic PFVP and their variable periods

The simplest functions with a variable period are trigonometric functions $\sin x^a$, $\cos x^a$, $x > 0, a > 0$. Fig. 2 shows the periodic function with a variable period $f_1(x) = \sin x^{3/5}$ (graph 1), and function $f_2(x) = \sin x$ (graph 2) for comparison. From the Fig. 2 we can see that with the argument x growing, the graph of $f_1(x) = \sin x^{3/5}$ stretches, i.e. its period increases. For the function $\sin x$ there are more than eleven periodic oscillations on the interval $[0,70]$, while for the function $\sin x^{3/5}$ on the same interval there are solely two oscillations.

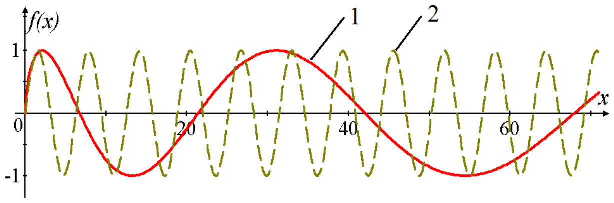

Fig. 2. Function $f_1(x) = \sin x^{3/5}$ (graph 1), $f_2(x) = \sin x$ (graph 2).

It is shown [4, 5] that for the sinusoidal functions with a variable period $\sin x^a$, $\cos x^a$, $a > 0$, their variable periods are determined by the formulas below:

$$T_\alpha(x) = -x + \left(x^\alpha + 2\pi\right)^{1/\alpha}, x \geq 0,$$
$$T_\alpha^{-}(t) = x - \left(x^\alpha - 2\pi\right)^{1/\alpha}, x \geq (2\pi)^{1/\alpha}. \quad (1)$$

Note that when there is no misunderstanding, the index a included in the expressions $T_a(x)$ and $T_a^{-}(x)$ can be omitted. Taking into account (1), the variable periods of $\sin x^{3/5}$, $x \ge 0$ are expressed by the following formulas:

$$T(x) = -x + \left(x^{3/5} + 2p \right)^{5/3},\ x \ge 0,$$

$$T^{-}(x) = x - \left(x^{3/5} - 2p \right)^{5/3},\ x \ge (2p)^{5/3} \approx 21.394.$$

Graphs of these periods are depicted in Fig. 3. For comparison purposes, $T = 2p$ for $\sin x$ is also given.

The graph of $T(x)$ confirms the behavior of the function $\sin x^{3/5}$ – with the values of the argument x increasing, the period $T(x)$ increases also. For example, at the point $x = 0$, $T(0) \approx 21.39438$, for $x = 30$, $T(30) \approx 51.12291$. Fig. 3 also shows that the period $T^{-}(x)$ decreases with the decrease of the argument.

It is not difficult to see that when $a > 1$, graphs of the functions $\sin x^{a}$, $\cos x^{a}$, $x > 0$, with the argument x growing, are compressed, therefore their period $T(x) = -x + \left(x^{a} + 2p\right)^{1/a}$, $x \ge 0$ is a decreasing function.

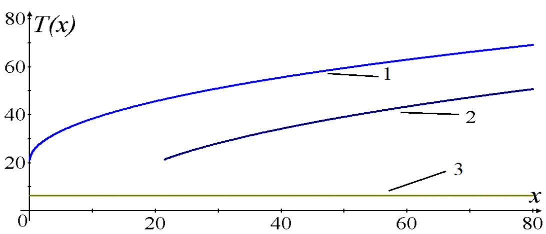

Fig. 3. Variable periods for the function $\sin x^{3/5}$, $x \ge 0$ *:*
$T(x) = -x + \left(x^{3/5} + 2p \right)^{5/3}$ *, (graph 1);* $T^{-}(x) = x - \left(x^{3/5} - 2p \right)^{5/3}$, $x \ge (2p)^{5/3} \approx 21.394$ *(graph 2). For comparison purposes, the period* $T = 2p$ *for the function* $\sin x$ *is also given (graph 3).*

4. Orthogonal system of trigonometric functions with a variable period

Like the system of trigonometric functions $\sin nx, \cos nx,\ n = 1.2.\mathbf{L}$, that are orthogonal on an arbitrary segment of length $2p$, there arises a natural question on the existence of orthogonal systems of trigonometric FVP. Some results on this issue were obtained in [2, 4]. In general, the following theorem is fulfilled.

Theorem. The system of trigonometric functions

$$\sin nx^{a},\ \cos nx^{a},\ x \ge 0,\ a > 0,\ n = 1.2,\mathbf{L}, \qquad (2)$$

with the variable period $T(x) = -x + \left(x^{a} + 2p\right)^{1/a}$, $x \ge 0$ is orthogonal with the weight function $r(x) = a x^{a-1}$ in the space $L_r^2(x, x + T(x))$, the norm of each function of system (2) being equal to $\sqrt{p}$.

We emphasize that for the space $L_r^2(x, x + T(x))$, the length of the orthogonality interval $[x, x + T(x)]$ is no longer constant, but varies according to the value of the period $T(x)$ at the point x. The orthogonality itself means that the scalar product of different functions of system (2) is equal to zero, and identical ones is equal to p, for example

$$\begin{aligned}
&\left(\sin mx^{a}, \sin nx^{a}\right) = \int_{t}^{t+T(t)} a\, x^{a-1} \sin mx^{a} \sin nx^{a}\, dx = 0, m \ne n,\\
&\left(\sin mx^{a}, \sin mx^{a}\right) = \int_{t}^{t+T(t)} a\, x^{a-1} \sin^{2} mx^{a}\, dx = p,\ m = 1, 2, \mathbf{L}
\end{aligned} \qquad (3)$$

The generalization of trigonometric FVP (TFVP) $\sin x^{a}$, $\cos x^{a}$, $x > 0$ is the functions $\sin g(x)$, $\cos g(x)$, $x \in I$, at the same time, the function $g(x)$, $x \in I$ must satisfy certain conditions [5], in particular, be continuous, piecewise differentiated, strictly increasing or decreasing, the domain of its definition is a certain interval $I = [a, b]$, and the variation satisfies the condition $V_a^b > 2p$. The variable period of functions $\sin g(x), \cos g(x)$ is denoted $T_g(x), x \in I$. Using the basic functions $\sin g(x), \cos g(x)$, we create a new system of trigonometric functions, which is the generalization of system (2):

$$\sin ng(x), \cos ng(x),\ x \in I,\ n = 1.2, \mathbf{L}. \qquad (4)$$

This system is orthogonal on the interval $[x, x + T_g(x)] \in I$ with the weight function $g'(x)$.

It is important to note that a partial case of system (4) is the Chebyshev polynomial. Let us consider a shortened version of system (4), specifically a system consisting solely of cos -

functions: $\cos ng(x)$, $x \in I, n = 1.2, \ldots$, and choose $g(x) = \arccos x$, $I = [-1,1]$. As a result, we obtain a system $\cos(n \arccos x), n = 1,2, \ldots$, which is orthogonal on the interval $I = [-1,1]$ with the weight function $g'(x) = (\arccos x)' = -1/\sqrt{1-x^2}$. The resulting TFVP system is a well-known system of Chebyshev polynomials of the first kind $T_n(x) = \cos(n \arccos x), n = 1,2, \ldots$.

5. Approximation algorithm

The presence of orthogonal systems of trigonometric FVP makes it possible to raise the question of the development of information technologies for doing research into PFVP. If we appeal to the recommendations of the constructive theory of functions [6, 7], then the direction of research, which is based on the "**algorithm of approximation**" is effective, especially for the applied use. Its essence is reduced to the replacement of the investigated function $f \in F$, where F is the set of functions approximated (in our case, the set of PFVP), by another function $j \in \Phi$, where Φ is the set of approximating functions. The approximating function j must satisfy certain requirements, in particular, be "close" to the approximation function f, reproduce its basic properties, contain sufficiently complete information about f, be comfortable in use. The Fourier series are preferably chosen as a set of approximating functions.

6. Fourier series of PFVP

As for the construction of a Fourier series for PFVP, let us also draw attention to the following issues. A variable period of the function for which a Fourier series is constructed and a variable period of the corresponding trigonometric system of functions must coincide. Let $f(x), x \ge 0$ be the function with the variable period $T(x) = -x + (x^a + 2p)^{1/a}$, $x \ge 0$. To construct its Fourier series, we use the TFVP system $\sin nx^a, \cos nx^a, a > 0, n = 1.2, \ldots$, whose variable period has the same analytical expression as the function period. Let us write the Fourier series $f(x) \approx a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx^\alpha + b_k \sin kx^\alpha$ for the function $f(x)$.

Taking into account scalar products (3), the Fourier coefficients of this series are determined by the formulas:

$$a_0 = \frac{a}{p} \int_t^{t+T(t)} x^{a-1} f(x)\,dx,$$
$$a_k = \frac{a}{p} \int_t^{t+T(t)} x^{a-1} f(x) \cos kx^a\,dx, \qquad (5)$$
$$b_k = \frac{a}{p} \int_t^{t+T(t)} x^{a-1} f(x) \sin kx^a\,dx.$$

The point $t \ge 0$ as the lower boundary of the integration interval is chosen arbitrarily. At this point, the variable period $T(t) = -t + (t^a + 2p)^{1/a}$, therefore the segment of integration is the orthogonality interval $[t, t + T(t)] = \left[t, (t^a + 2p)^{1/a}\right]$. Bessel's inequality has the form

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2 \le \frac{1}{p} \| f(x) \|^2, \qquad (6)$$

where $\|f\| = \sqrt{(f, f)} = \sqrt{a \int_x^{x+T(x)} x^{a-1} f^2(x) dx}$.

Let us consider an example of calculating the coefficients and constructing a Fourier function for $f(x) = sign(\sin x^a), x \ge 0$, with the variable period $T(x) = -x + (x^a + 2p)^{1/a}$ when $\alpha = 3/5$, and, for comparison purposes, carry out similar calculations for the same function when $a = 1$, i.e. for the well-known from mathematical analysis function $f(x) = sign(\sin x)$, which describes periodic oscillations of rectangular form with the constant period $T = 2p$. Let us also check Bessel's inequality for these functions.

Example. $T(x) = -x + \left(x^{3/5} + 2p\right)^{5/3}$ $x \ge 0$ is the variable period for $f(x) = sign\left(\sin x^{3/5}\right), x \ge 0$ (Fig. 4). To calculate the Fourier coefficients, it is necessary that the integration be performed according to formulas (5) on the interval $[t, t + T(t)] = \left[t, \left(t^{3/5} + 2p\right)^{5/3}\right]$. Let its left point be $t = 20$ for this interval. At this point $T(20) = -20 + \left(20^{3/5} + 2p\right)^{5/3} \approx 45.69457$, therefore the integration interval is the interval $[20,\ 20 + T(20)] \approx [20,\ 65.69457]$.

The resulting first five coefficients are given in Table 1. For comparison purposes, Table 2 represents the Fourier coefficients of the function $f(x) = sign(\sin x)$ calculated according to the same formulas (5), but with $a = 1$.

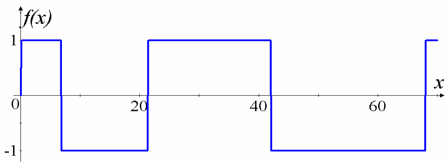

Fig. 4. Graph of the function $f(x) = sign\left(\sin x^{3/5}\right)$.

The comparison of the Fourier coefficients for the functions $f(x) = sign\left(\sin x^{3/5}\right)$ and $f(x) = sign(\sin x)$ shows their "practical" coincidence. Some inconsistencies can be explained by errors in calculations.

Table 1

Fourier coefficients of a function with variable period

Coefficient number	Function with variable period $f(x) = sign\left(\sin x^{5/7}\right)$	
	Orthogonality interval $[20,\ 65.69457]$	
	$a_k,\ k = \overline{0,10}$	$b_k,\ k = \overline{1,10}$
0	0.00005	
1	0.00004	1.27324
2	0.00005	0
3	-0.00001	0.42441
4	0.00003	0
5	-0.00001	0.25465
	$\frac{a_0^2}{2} + \sum_{k=1}^{9} a_k^2 + b_k^2 = 1.9797632$	
	$\frac{1}{p}\left\Vert f(x)\right\Vert^2 = 2$	

Table 2

Fourier coefficients of a function with constant period

Coefficient number	Function with constant period $f(x) = sign(\sin x)$	
	Orthogonality interval $[0, 2p] = [0,\ 6.28319]$	
	$a_k,\ k = \overline{0,10}$	$a_k,\ k = \overline{0,10}$
0	0	
1	0	1.27324
2	0	0
3	0	0.42441
4	0	0
5	0	0.25465
	$\frac{a_0^2}{2} + \sum_{k=1}^{9} a_k^2 + b_k^2 = 1.97974$	
	$\frac{1}{p}\left\Vert f(x)\right\Vert^2 = 2$	

Using the obtained Fourier coefficients, we construct finite Fourier series of these functions. Fig. 5 shows a graph of the finite Fourier series of $f(x) = sign\left(\sin x^{3/5}\right)$ (solid line) and a graph of the function itself (dotted line). Comparing these graphs, we can argue that with $n = 20$ the finite Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx^{3/5} + b_k \sin kx^{3/5}$$

is "good" to reproduce the form of the function itself. Similar conclusions can also be drawn while comparing the finite Fourier series both for the function $f(x) = sign(\sin x)$, whose graph is shown in Fig. 6 (solid line), and for the function $f(x) = sign(\sin x)$ (dotted line).

The Tables also present the calculation results for the formulas included in Bessel's inequality (6). The comparison shows that the calculation results for these functions also practically coincide. This suggests that the Fourier series of PFVP constructed in the work is sufficiently close to the function itself.

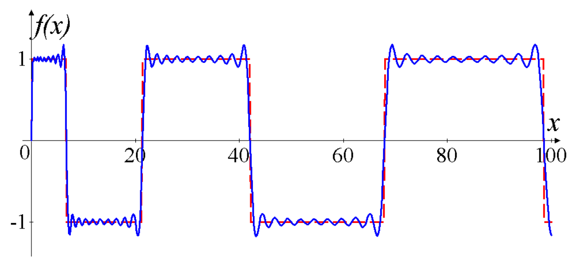

Fig. 5. Function $f(x) = sign\left(\sin x^{3/5}\right)$, $x \in [0,100]$, *(dotted line) and its Fourier series (first forty terms, solid line).*

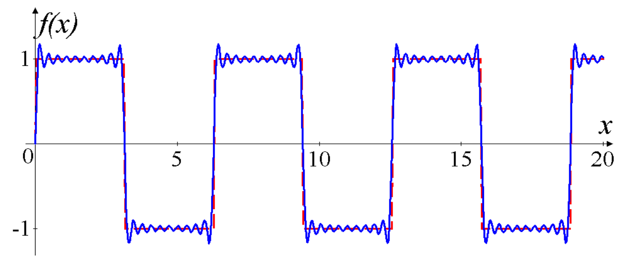

Fig. 6. Function $f(x) = sign(\sin x)$, $x \in [0,20]$, *(dotted line) and its Fourier series (first forty terms, solid line).*

The information technologies developed for the construction of the Fourier series for PFVP will form the basis for the research both into analytically given functions and into empirical ones, primarily electrocardiograms with a variable period.

7. The heart rhythm variability

In the applied research into the PFVP, knowledge of the VP is rather an exception than regularity, therefore we often deal with the task of finding its evaluation. A similar situation is observed for periodic functions and for periodic random processes with a constant period. But if the tasks and methods of evaluating the constant period were considered in a large number of works, as indicated, for example, in [8], the issue of evaluating the variable period remains unstudied.

We can indicate two important directions for using the variable period evaluation. The availability of an evaluation allows us to consider the problems of constructing Fourier series of the PFVP, using the evaluation of the variable period instead of its exact value. In addition to the construction of the Fourier series, the evaluation of the variable period will have an important significance for the problems of its direct use, primarily in cardiology, in the study of the cardiac rhythm variability. According to [9], the heart rhythm variability (HRV) is unevenness in the frequency of heart contractions caused by the influence of various regulatory processes in the human body.

The question of variability is considered in many works of medical and statistical direction. There are several methods for evaluating the heart rhythm variability. Statistical methods form an important group. The basis of these studies is the methods for analyzing the sequence of $R-R$ cardiogram intervals, i.e. the intervals between adjacent heart contractions. The literature review shows that the methods of statistical analysis are based on the assumption (hypothesis) that the specified sequence is stationary. Scientifically grounded methods of studying variability for cases where the sequence of $R-R$ intervals is different from the stationary one are practically lacking. And only the study of PFVP, as well as solving the problem of evaluatingthe variable period open the possibility of developing a theory and methods for studying the variability of the heart rhythm in nonstationary regimes, i.e. in the cases of impact of some “exciter of calm”, for example, physical exertion, on the human body.

8. Evaluation of VP and cardiac rhythm variability after physical activity

Consider an example of VP evaluation of an electrocardiogram (ECG), obtained after an impact of physical exertion, on the patient. and determination of diagnostic parameters of variability. At the first stage of the experiment, the patient was subjected to physical activity (twenty deep squats). Immediately after squatting the $R-R$ intervals were selected, i.e. the values of its VP $T(t)$ at the time points $t_k, k=1,2,\mathbf{L},n$: $T_k=t_{k+1}-t_k$, where t_k – the moment of the appearance of the k-th R-teeth of the ECG. On the basis of the values of the VP T_k, the values of the variable frequency (VF) $\nu_k = 1/T_k$ were calculated. Each tenth value T_k and n_k is shown in Fig. 7, graphs 1b and 2b. The analysis of the dynamics of the change in T_k and n_k shows that it is initially expedient to carry out the approximation of the VF n_k, since for values n_k exponential dependence of the form $n(t)=a+be^{-at}, t\in(0,\infty)$ is characteristic.

Using the values n_k and mathematical package [10], the coefficients a, b and the parameter a were calculated using the least squares method: $a=1{,}176$, $b=0{,}995$, $a=0{,}014$. Graphs of VF $n(t)=a+be^{-at}=176+0.995\times e^{-0.014}$ and VP $T(t)=1/\left(1.176+0.995\times e^{-0.014}\right)$ are shown in Fig. 7 (graphs 1a and 2a, respectively).

Note that from the standpoint of cardiological analysis, the parameter a means the heart rate (heart contractions), that occurs in some time after its stabilization (resting heart rate), the parameter b indicates the magnitude of the increase in the heart rate in comparison with the frequency in the resting state, the parameter a characterizes the "speed" of pulse stabilization.

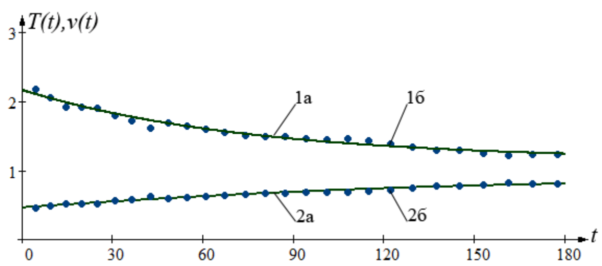

Fig. 7. Experimental values of the VF n_k and the VF function $n(t)$ (graphs 1a and 1b), the value of the VP T_k and the function of the VP $T(t)$ (graphs 2a and 2b).

In addition to the parameter a that is a characteristic of the speed of pulse stabilization, the duration of the stabilization, i.e. the interval from the moment t_1, when the counting of the $R-R$ intervals T_k began, until the moment $\hat{t}$, when the pulse has stabilized, is informative in the diagnostic tasks. The value $\hat{t}$ can be found as such a moment on the numerical axis $(0,\infty)$, when the difference $n(\hat{t})-a=e>0$. At the same time for $t<\hat{t}$, the difference $n(t)-a>e$, for $t>\hat{t}$, the difference $n(t)-a<e$. The value e is chosen based on practical considerations and advisory advice from cardiologists.

9. Evaluation of post-exercise cardiac rhythm arrhythmia

According to many literary sources of medical nature, an important indicator of cardiac rhythm variability is arrhythmia, that is, the mean square deviation of the values of the sequence of $R-R$ intervals from the evaluation of their mathematical expectation. When calculating the value of arrhythmia, the sequence of $R-R$ intervals is considered stationary [9]. In our case, i.e. after the impact of physical exertion on the human body, the sequence of $R-R$ intervals is significantly different from the stationary one, so the method for the determination of arrhythmia described above is not appropriate.

To evaluate the post-exercise cardiac arrhythmias, we propose the aboveused algorithm, which takes into account the deviations of the $R-R$ intervals, that is, the values $T_k,\ k=1,2,...,n$ from VP $T(t)=1/v(t)=1/\left(a+be^{-at}\right)$. By the amount of arrhythmias for the case described, it is natural to use the value

$$S_T^2=\sum_{k=1}^{n}(T_k-T(t_k))^2=\sum_{k=1}^{n}\left(T_k-\frac{1}{a+be^{-at_k}}\right)^2 \quad (7)$$

or the corresponding mean square deviation

$$\sigma_T=\sqrt{S_T^2/n}\,. \quad (8)$$

There is no doubt that for practice problems, it is expedient to use a mean square deviation s_T. It is also necessary to note that in formula (7) the values $T_k,\ k=1,2,\mathbf{L},n$ are the values of $R-R$ intervals (of the variable period) obtained as a result of the experiment, $T(t_k)$ – the values of the variable period, calculated by the formula

$$T(t)=1/\left(1.176+0.995\times e^{-0.014}\right)$$

in points $t=t_k$.

In addition to evaluating arrhythmias that are calculated using $R-R$ intervals, the "arrhythmia" of the heart contractions frequency can be calculated similarly. Quadratic arrhythmia (deviation) of frequency

$$S_u^2=\sum_{k=1}^{n}(u_k-u(t_k))^2=\sum_{k=1}^{n}\left(u_k-\left(a+be^{-at_k}\right)\right)^2, \quad (9)$$

mean square arrhythmia of frequency

$$\sigma_v=\sqrt{S_v^2/n}\,.$$

The results of the above experiment (experiment 1), that is, the values of the coefficients a, b, the parameters a, s_T and s_u, are given in the table. For comparison, the results of another experiment (experiment 2) are also given.

Table 3

Experi-ment number	a	b	a	s_T	s_u
1	1.176	0.995	0.014	0.025	0.017
2	1.127	1.121	0.020	0.027	0.018

The values of the coefficients a, b and the parameter a, included in the formula VP $T(t)$ and VF $n(t)$, and the values of the parameters s_T and s_u are proposed to use as new diagnostic features of the variability of the post-exercise heart rhythm.

7. Conclusions

It is noted that in addition to empirical periodic signals, whose model is a periodic function, applied research deals with signals also characterized by periodicity, but their period changes in some way. However, if for periodic functions, their theory, first of all, the Fourier series, and analysis methods have been developed quite deeply, signals with a variable period practically have not been studied. The paper highlights the main achievements of the theory and methods of studying signals with a variable period. The basis of the research is the model of such signals – these are periodic functions with a variable period. The main direction of the research is the recommendations of the constructive theory of functions – approximation of functions with a variable period by the Fourier series. To construct such series, we have developed methods for the formation of orthogonal systems of trigonometric functions with a variable period, with their variable period having to coincide with the period of the investigated function. An example of constructing a Fourier series for the analytically given function with a variable period is considered. The issue of evaluating the variable period of the PFVP is also raised, and for the ECG obtained after physical activity, evaluations of the VF and VP were derived. The evaluation of the VF turned out to have the form of exponential function, which is determined by three parameters.The obtained results reveal the perspective directions of the research into real signals with a variable period, in particular, electrocardiograms obtained during or after the exposure to "the exciter of calm", and the values of the parameters of the VF evaluation significantly complement the arsenal of diagnostic features of the ECG and VF of heartbeat.

References

[1] M. Pryimak, I. Bodnarchuk, and S. Lupenko, "Conditionally-periodic stochastic processes with variable period", *Visnyk Ternopilskoho derzhavnoho tekhnichnoho universytetu,* no. 2, pp. 143–152, 2005. (Ukrainian)

[2] M. Pryimak, "Periodic functions with variable period", 2010, *http://arxiv.org/pdf/1006.2792v1*

[3] M. Pryimak, R. Sarabun, and L. Dmytrotsa, "Evaluation of variable period and frequency", *Vymiryuvalna ta obchyslyuvalna tekhnika v tekhnolohichnykh protsesakh,* Khmelnytskyi, Ukraine, no. 2, pp. 76–82, 2011. (Ukrainian)

[4] M. Pryimak, Ya. Vasylenko, and L. Dmytrotsa, "Signals with variable period and their simulation", *Visnyk NTUU "KPI", Seriya "Informatyka, upravlinnya ta obchyslyuvalna tekhnika",* Kyiv, Ukraine, no. 59, pp. 116–121, 2013. (Ukrainian)

[5] Ya. Vasylenko, L. Dmytrotsa, M. Oliynyk, and M. Pryimak, "Methods of description of functions with variable period and their approximation", *Visnyk Kharkivskoho natsionalnoho universytetu imeni V. N. Karazina,* pp. 36–47, 2016. (Ukrainian)

[6] S. Bernshtein, *Constructive theory of functions.* Moscow, Russia: Academy of Sciences, vol. 2, 1954. (Russian)

[7] I. Natanson, *Constructive theory of functions.* Moscow, Russia: Gos. Izd-vo tekhniko-teoreticheskoy literatury, 1949. (Russian)

[8] M. Serebrennikov and A. Pervozvanskiy, *Separation of hidden periodic functions.* Moscow, Russia: Nauka, 1965. (Russian)

[9] S. Kovalenko and L. Kudii, *Variability of heart rhythm. Methodology*. Cherkasy, Ukraine: Bohdan Khmelnytskyi National University, 2016. (Ukrainian)

[10] D. Kiryanov, *Mathcad 11.* St, Petersburg, Moscow: BHV-Peterburg, 2003. (Russian)

ПЕРІОДИЧНІ ФУНКЦІЇ ЗІ ЗМІННИМ ПЕРІОДОМ ТА ЇХ НАБЛИЖЕННЯМИ РЯДАМИ ФУР'Є

Микола Приймак, Ярослав Василенко, Леся Дмитроца, Марія Олійник

Звернуто увагу, що крім періодичних емпіричних сигналів, моделлю яких є періодичні функції, трапляються сигнали, які ведуть себе подібно періодичним, але при цьому період повторюваності їхніх значень уже не є постійним, а певним чином змінюється. Наглядним прикладом є електрокардіограми (ЕКГ), отримані під час чи після дії на організм пацієнта певного збудника спокою, наприклад, фізичного навантаження. Яким же чином досліджувати періодичні сигнали із змінним періодом (ПСЗП)? Огляд літературних джерел показує, що донедавна на це питання будь-якої науково обґрунтованої відповіді не було. Тому розроблення інформаційних технологій (ІТ) дослідження ПСЗП є актуальною як з теоретичного, так і прикладного погляду. Для її вирішення пропонують використовувати підхід, суть якого вкладається в тріаду "модель–алгоритм–програма". Певні результати на цьому шляху вже отримані в попередніх роботах авторів цієї статті, зокрема наведено означення періодичної функції із змінним періодом (ПФЗП), розглянуто приклади тригонометричних ФЗП (ТФЗП) та записані їхні змінні періоди, розроблено метод утворення системи ортогональних ТФЗП і визначено скалярний добуток для функцій системи. У цій роботі записано ряд Фур'є для ПФЗП та отримано формули для знаходження його коефіцієнтів. Як приклад побудовано скінчений ряд Фур'є для аналітично заданої ПФЗП та показано, що із збільшенням кількості коефіцієнтів ряд наближається до самої функції, що підтверджує правильність отриманих теоретичних результатів.

Враховуючи, що для більшості емпіричних ПФЗП їх змінний період є невідомим, порушено питання його оцінки. Для випадку ЕКГ, отриманої після фізичного навантаження, побудовані оцінки її змінної частоти (ЗЧ) та ЗП. Виявилося, що оцінка ЗЧ має вигляд експоненційної функції, що визначається трьома параметрами. Розроблені ІТ для вивчення ПФЗП дають можливості досліджувати реальні ПСЗП, зокрема ЕКГ зі ЗП, а отримані числові значення параметрів використовувати в задачах діагностики, підтримки прийняття рішень.

Mykola Pryimak – D. Sc., Prof. Graduated from Ivan Franko State University, Lviv, Ukraine (major “Computing Mathematics”). He defended his D.Sc dissertation thesis in Information Technology at National Aviation University, Kyiv, Ukraine. Since 2001 he has held the post of Head of Department of Computer Sciences, Ternopil Ivan Puluy National Technical University. Areas of research: information technology, the laws of the material and spiritual worlds.

Yaroslav Vasylenko – graduated from Taras Shevchenko State University of Kyiv, attended postgraduate studies at the Institute of Mathematics of the National Academy of Sciences of Ukraine. He works at Ternopil Volodymyr Hnatiuk National Pedagogical University as Assistant Professor at the Department of Computer Sciences and Teaching Methodology. Areas of research: mathematical modelling and information technology.

Lesia Dmytrotsa – graduated from Ivan Puliuy State Technical University (major “Instrument making”), Ternopil, Ukraine. She works as Assistant Professor at the Department of Computer Sciences, Ivan Puliuy National Technical University of Ternopil, Ukraine.

Areas of research: mathematical modelling and computational methods, information technology, computer systems of processing of various kinds of information.

Mariya Oliynyk – graduated from Lviv Polytechnic Institute, Ukraine. She works as Engineer at the Department of Computer Sciences of Ivan Puliuy National Technical University of Ternopil, Ukraine. Area of research: mathematical modelling and computational methods.