GEODESY

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Alexander N. MARCHENKO[1], Serhii PERII[2], Ivan POKOTYLO[3],
Zoriana TARTACHYNSKA[4]

[1-4]Lviv Polytechnic National University, Institute of Geodesy, 6, Karpinsky, Str., Lviv, 79013, Ukraine,
e-mail: march@pancha.lviv.ua; periy_ss@ukr.net; ivan.y.pokotylo@lpnu.ua; ztartachynska@yahoo.com,
[1]https://orcid.org/0000-0002-9315-5903, [2]https://orcid.org/0000-0002-2489-3275,
[4]https://orcid.org/0000-0002-5704-6128

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GRAVITATIONAL POTENTIAL ENERGY AND FUNDAMENTAL PARAMETERS OF THE TERRESTRIAL AND GIANT PLANETS

The basic goal of this study (as the first step) is to collect the appropriate set of the fundamental astronomic-geodetics parameters for their further use to obtain the components of the density distributions for the terrestrial and outer planets of the Solar system (in the time interval of more than 10 years). The initial data were adopted from several steps of the general way of the exploration of the Solar system by iterations through different spacecraft. The mechanical and geometrical parameters of the planets allow finding the solution of the inverse gravitational problem (as the second stage) in the case of the continued Gaussian density distribution for the Moon, terrestrial planets (Mercury, Venus, Earth, Mars) and outer planets (Jupiter, Saturn, Uranus, Neptune). This law of Gaussian density distribution or normal density was chosen as a partial solution of the Adams-Williamson equation and the best approximation of the piecewise radial profile of the Earth, including the PREM model based on independent seismic velocities. Such conclusion already obtained for the Earth's was used as hypothetic in view of the approximation problem for other planets of the Solar system where we believing to get the density from the inverse gravitational problem in the case of the Gaussian density distribution for other planets because seismic information, in that case, is almost absent. Therefore, if we can find a stable solution for the inverse gravitational problem and corresponding continue Gaussian density distribution approximated with good quality of planet's density distribution we come in this way to a stable determination of the gravitational potential energy of the terrestrial and giant planets. Moreover to the planet's normal low, the gravitational potential energy, Dirichlet's integral, and other planets' parameters were derived. It should be noted that this study is considered time-independent to avoid possible time changes in the gravitational fields of the planets.

Key words: fundamental astronomic-geodetics parameters; solution of the inverse gravitational problem; Gaussian density distribution; Dirichlet's integral.

Introduction

One of the basic papers regarding the internal structure of the Solar planets has been written by the pioneer of geophysics Georg H. Darwin in (1877) where their some fundamental parameters have been adopted for the estimation of the internal densities of terrestrial and outer planets. According to the "Encyclopedia of the Solar System" (2015) most large bodies of the Solar system have been discovered much earlier before the space age of Planetary Exploration. The study of planets through space probes usually contains several basic steps by iterations.

In the first stage, the flybys give close images and measure certain physical properties of the planets. Further use of satellites around orbits close to the celestial bodies leads to mapping most of the surface and taking detailed remote sensing measurements. In the following the exploration continues with landing a probe on the surface to make in situ measurements. Returning samples to the Earth for detailed analysis are the next step before a human landing on a planetary system might be considered. Considering these steps we come again to the general way of the exploration of the Solar system by iterations.

In reality, all the planets and some smaller bodies have been visited by spacecraft, but only the Moon was studied, including all the above-mentioned steps, including the additional system of two low-low GRAIL satellites with a minimum altitude of 23 km, which allows the detailed study of the gravity field [Konopliv, et al., 2014; Lemoine, et al., 2014] and the topography [Zuber, et al., 2016] of the Moon with a horizontal resolution of 3–5 km. A summary of the exploration steps of the large objects in the solar system is presented by flybys and orbiting missions to the Earth, Mercury (Mariner 10, 1974; MESSENGER, 2008), Venus (Mariner 2, 1962; Venera 9, 1975), Mars (Mariner 4, 1965; Mariner 9, 1971; Viking1, 1976), Jupiter (Pioneer 10, 1973; Pioneer 11, 1974; Voyagers 1, 1979; Voyagers 2, 1979; Galileo, 1995; JUNO mission, 2016), Saturn (Pioneer 11, 1979; Cassini 2017), Uranus (Voyager 2, 1986), and Neptune (Voyager 2, 1986). Therefore, Uranus and Neptune have not been explored from space in detail apart from the single Voyager 2 flybys.

Thus, because many key characteristics of planets regarding their origin, evolution, and internal structure remain unresolved, the problem of clarifying the fundamental parameters of the translation – rotational motion of planets, their use to build radial density profiles, including estimates of such an important parameter as the gravitational potential energy of planets.

First descriptions of the Earth’s density in the form of spherically symmetric density distributions were investigated by Legendre, Laplace, Darwin, Roche, etc. Recently, instead of the polynomial representation of a piecewise radial density [Dziewonski and Anderson, 1981], one hypothesis was analysed especially in view of the Williamson-Adams equation [Bullen, 1975]. The latter leads to the famous Gaussian distribution that was called by the Earth’s density normal law [Marchenko, 2000]. Fig. 1 illustrates several continuous density distributions including PREM piecewise radial density.

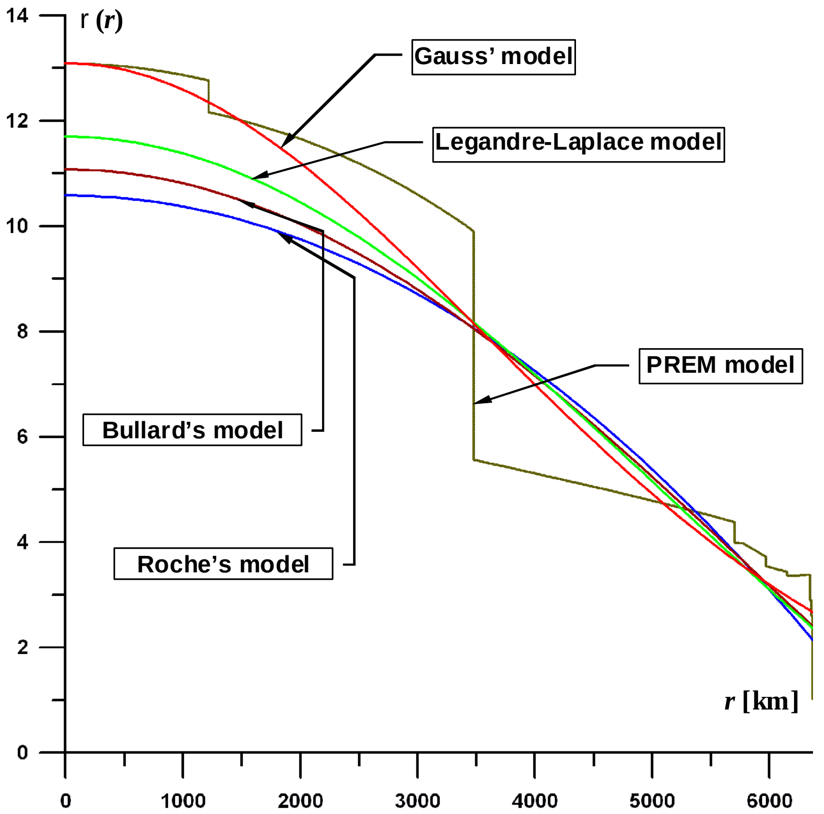


Fig. 1. The Earth Legendre–Laplace, Roche, Bullard, and Gauss continuous densities compared with piecewise PREM-density model r (l), g/cm^3

It should be noted that the case of the Earth the Gaussian distribution of density (or the normal law) agrees best of all with the PREM-model also based on the seismic data. On the other hand, the continues Gauss’ model of radial density leads to the most reliable estimation E_{Gauss} of the potential energy E [Marchenko, 2009] taken from the piecewise PREM-model [Marchenko, & Zayatz, 2011] answering the following question: what kind of continuous density law should be chosen to compare such general characteristics as the mass of the planet, the average moment of inertia, or the gravitational potential energy of the planet. Hereafter the energy E taken with the sign $(-)$ is the work $W(=-E)$ of gravitation required to transport the masses, having the total Earth’s mass *M*, “from a state of infinite diffusion to its actual condition" inside the planet [Thomson, & Tait, 1883]. Gauss proved in his famous memoir (1867) that $W = -E$ has some minimal value $W_{\min}$ if all masses are concentrated on the planet's boundary S considered as a level surface.

Therefore, this study focuses on the estimation $E = -(W_{\min} + \Delta W)$ for the different planets of a terrestrial kind and giant planets based only on the Gaussian density or normal distributions. The well-known formula allows a simple search for $W_{\min}$ and offers an important interpretation of the deviation ΔW from $W_{\min}$ which dependences on Dirichlet's integral applied on the internal potential V_i generated from an adopted density distribution. It should be noted that this study is considered as time-independent to avoid possible time changes in the gravitational fields of the planets.

Initial data. Adams–Williamson equation

The density ρ may be satisfying to the so-called Adams–Williamson equation for each shell of the stratified Earth under the following assumptions ($0 \le \lambda \le R$) where R is the mean radius of the spherical planet: 1) the Earth is globally in hydrostatic equilibrium $\frac{\partial \rho(\lambda)}{\partial \lambda} \le 0$; 2) chemical composition and phase transformation are homogeneous in every shell; 3) the temperature is adiabatic in each shell. In the integral form the condition of the hydrostatic equilibrium requires a minimum of gravitational energy. Thus, if we observe seismic velocities V_p and V_s in the form of the function

$$\Phi = \Phi(\lambda) = V(\lambda)_P^2 - \frac{4}{3} V(\lambda)_S^2 , \qquad (1)$$

in view of the gravitational

$$g(\lambda) = \frac{4\pi G}{3} \lambda \cdot \delta_m(\lambda) \quad \Leftrightarrow \quad g(\lambda) = \frac{GM}{\lambda^2} , \qquad (2)$$

and hydrostatic relationships

$$grad\,p(\lambda) = \rho(\lambda) \cdot grad\,V(\lambda) \quad \Rightarrow \quad \frac{dp(\lambda)}{d\lambda} = -\rho(\lambda) \cdot g(\lambda) , \qquad (3)$$

finally the Adams–Williamson equation can be written as

$$\frac{d \ln \rho(\lambda)}{d\lambda} = -\frac{g(\lambda)}{\Phi(\lambda)} , \qquad (4)$$

where p is the pressure inside the Earth. Thus (4) represents the formula to derive the radial density distribution from the seismic velocity data, fulfilled under the assumptions above. In order to use (4) we must first try to solve this equation and to express the observed seismic data by a suitable function of depth, separating the Earth into convenient shells. Traditionally we shall assume that the separation into shells has to be choice at those spheres, where discontinuities in the parameter Φ or in its derivative can be observed.

Table 1

Fundamental parameters of the planets (All NASA parameters were updated in 2020)

Planet	a_e, km	ρ_m, g/cm^3	I (or C)	R	ρ_s, g/cm^3
1	2	3	4	5	6
Terrestrial planets					
Mercury	2439.4 Messenger **2439.5 NASA**	**5.428** Margot, et al. 2012 5.427 NASA	**C=0.346 ±0.014 Margot, et al. 2012**: I=0.35 NASA	**2439.7 NASA**	**2.9 Rivoldini, et al., 2009**
Venus	**6051** Magellan	5.25 Cottereau, Souchay, 2009 **5.243** NASA	C=0.3360 Cottereau, Souchay, 2009 **I=0.33 NASA**	**6051.8 NASA**	2.85 Yoder, 1995 **2.9 Rappaport, et al., 1999**
Earth	**6378.137 NASA**	**5.514 NASA**	**0.3308 NASA**	**6371 NASA**	**2.67 Moritz, 1990**
Moon	**1738 GRAIL**	**3.344 NASA**	0.394 **NASA**	**1737.4 NASA**	**2.8 Hikida**, **Wieczorek, 2007**

Continuation of Table 1

1	2	3	4	5	6
Mars	**3396** Mars Reconnaissance Orbiter	3.935 ±0.4 Seidelmann, et al., 2002 **3.933 NASA**	0.3653 Mocquet, et al., 2010 **0.3638 Konopliv, et al., 2016**	**3389.5 NASA**	3.0 Zuber, et al., 2000; **2.9 Konopliv, et al., 2006**
Outer planets					
Jupiter	71492 **NASA**	**1.326 NASA**	**C=0.2629 – 0.2645** Helled, et al., 2011 **0.254 NASA**	**69911 NASA**	**<0.8 NASA**
Saturn	60268 **NASA**	**0.687 NASA**	**C=0.218** Helled, et al., 2011 **0.210 NASA**	**58232 NASA**	**<0.4 NASA**
Uranus	**25559 NASA**	**1.271 NASA**	**0.225 NASA**	**25362 NASA**	**<0.1 NASA**
Neptune	**24764 NASA**	**1.638 NASA**	**0.230 NASA**	**24622 NASA**	**<0.1 NASA**

[a_e is the equatorial radius; R is the mean radius; ρ_m is the mean density; I is the dimensionless mean moment of inertia; ρ_s is the surface density. All NASA planet's parameters were taken from the web-site (http://nssdc.gsfc.nasa.gov/planetary/). All adopted parameters are highlighted in bold text.]

It is evident that the formal solution of (4) may be obtained after the integration of Adams–Williamson equation. The result is

$$\rho(\lambda)=\rho_0\exp\left(-\int_0^\lambda \frac{g(x)}{\Phi(x)}dx\right), \qquad (5)$$

and we get the functional dependence for radial density as the exponential function. The right hand side of the expression (5) is unknown. For this reason, we shall apply instead of (5) the simplest approximating function

$$\rho(\lambda)=\rho_0\exp\left(-\beta^2x^2\right),\ \beta = \text{const}, \qquad (6)$$

where the degree 2 is the lowest power for which we get a non-zero value Φ at the origin.

Table 1 demonstrates adopted fundamental parameters of the planets derived from the exploration of the Solar system and applied in the following as initial information. For the applications, we shall write some well-known formulas within the sphere of the radius λ (the part of the Earth's mass which is restricted by this radius) for the mass

$$M(\lambda)=4\pi\int_0^\lambda \rho(x)x^2dx, \qquad (7)$$

where dx is the element of a line and the mean density $\rho_m(\lambda)$:

$$\rho_m(\lambda)=\frac{3}{4\pi\lambda^3}M(\lambda). \qquad (8)$$

The value $\rho_m(\lambda)$ in the form of (8) leads to the above-mentioned representations (2). The expression (6) admits according to (1) and (4) the next remarkable expressions for the mass

$$M(\lambda)=\frac{4\pi\rho_0R^3}{\beta^2}\left[\frac{\sqrt{\pi}\cdot \mathrm{erf}(\beta\cdot x)}{4\beta}-\frac{x}{2\rho_0}\rho(\lambda)\right], \qquad (9)$$

and for the mean moment of inertia

$$I(\lambda)=\frac{8\pi\rho_0R^5}{3\beta^4}\left[\frac{3\sqrt{\pi}\cdot \mathrm{erf}(\beta\cdot x)}{8\beta}-\frac{x}{4\rho_0}\rho(\lambda)\cdot\left(2\beta^2x^2+3\right)\right]=\frac{R^2}{\beta^2}\left[M(\lambda)-\frac{4\pi\lambda^3}{3}\rho(\lambda)\right], \qquad (10)$$

where $erf(z)$ is the integral of the Gaussian distribution from 0 to z or the probability integral with the density distribution according to (6).

Thus we come to a remarkable result: 1) one of the solutions of the Adams-Williamson equation is nothing else but the famous Gaussian (normal) distribution, which may be approximated by Roche's model, represented the possible solution of the Clairaut equation; 2) in the case of the Gaussian density distribution, the mean moment of inertia (with the accuracy to the certain constant) can be expressed as the simple difference between the Earth's mass and the mass of the homogeneous Earth with such mean density, which is equal to its surface density $\rho(\lambda)$.

Gauss' distribution of the planets density

According to [Moritz, 1990] three above-mentioned conditions had lead to the following conclusion "any global density law must satisfy three basic conditions" [see also Bullard, 1954]: 1) It must provide the total mass or, equivalently, the mean density; 2) It must give the value for the mean moment of inertia; 3) It must reproduce the density at the base of continental layers, which may be taken as about 3.2 to 3.3 g/cm^3, e.g. the conventional density just below Mohorovichich discontinuity much used in isostasy ρ_M =3.27 g/cm^3 ".

These three conditions may lead to the construction of the continuous radial density distribution. The first two conditions can apply for the determination of the continuous Roche's model. First, according to Gauss' model for the density at the origin, which is dependent on the observed value of Φ we get

$$\Phi_1(0) = -\frac{2\cdot\pi\cdot G\cdot R^2}{3\cdot\beta_1^2}\rho_0, \qquad (11)$$

where $\Phi_1(0)$ corresponds to the first piece of the seismic data Φ at the origin. According to the above stratification we get (i=1, 2, … m) pieces on the whole. Next, after simple manipulations, the following formula may be written

$$\beta_m = -\frac{2\cdot\pi\cdot G\cdot R^2\rho_0}{3\cdot\Phi_1(0)}\cdot\frac{\Phi_1(\lambda_1)\cdot\Phi_2(\lambda_2)\cdot\Phi_3(\lambda_3)...\Phi_{m-1}(\lambda_{m-1})}{\Phi_2(\lambda_1)\cdot\Phi_3(\lambda_2)\cdot\Phi_4(\lambda_3)...\Phi_m(\lambda_{m-1})}, \qquad (12)$$

which leads to the following conclusion If the density of the Earth is known at the origin, then the seismic data at the boundaries of the Earth's jumps is sufficient to independently determine the coefficients (12) of the piecewise Gaussian model..

However, according to (12) we can add an additional density condition to the origin, which will depend on the observed value Φ. In addition to the previous results and conclusions, we will finally describe some simplest properties of the normal model, having the form of a spherical shell with the corresponding stratification of the planet and return to the three conditions listed at the beginning of this section. After the transformation (10) to the dimensionless moment of inertia $I_m = I/MR^2$, we obtain the simplest relation

$$I_m = \frac{1}{\beta^2}\left[1-\frac{\rho_s}{\delta_m}\right], \qquad (13)$$

between the Earth's mean density δ_m, the moment of inertia I_m, and the surface density ρ_s. This formula provides for our non-linear inverse problem the remarkable closed expression

$$\beta^2 = \frac{1}{I_m}\left[1-\frac{\rho_s}{\delta_m}\right], \qquad (14)$$

for the coefficient β of the continuous Gauss' density model.

Thus, we come to the sequential solution of the non-linear inverse problem by means of two closed expressions for 2 basic parameters. The first one is the *qualitative characteristic* (14) of the global density distribution.

The second one is the *quantitative characteristic*

$$\rho_0 = \frac{4\cdot\beta^3\rho_m\cdot\exp(\beta^2)}{3\cdot\left(\sqrt{\pi}\cdot\exp(\beta^2)\cdot\mathrm{erf}(\beta)-2\beta\right)}, \qquad (15)$$

which was derived from the expression (8) for the Gauss' mean density. The solution of (14)–(15) provides finally the density at the centre mass of the planet, and a remarkable agreement of ρ_s, the mean density ρ_m, and the mean moment of inertia I_m. In the case of the Earth this model agrees best of all with the continuous Bullard's model (see Fig. 1), which has 3 parameters.

It should be noted that the determination of 2 parameters of the continuous Roche's model is based only on the mean density ρ_m and the mean moment of inertia I_m, without any condition for ρ_s. As a matter of fact, in the case of the continuous Gauss' density model with 2 parameters all these 3 conditions can be replaced by one relationship (14) for the qualitative characteristic β of the planet's normal density distribution. The quantitative characteristic ρ_0 now is the non-linear function (15) of the computed β and the Earth's mean density.

Table 2

Parameters of the normal law and Roche's model for the Planets of Solar system

Planet	ρ_0, g/cm^3	β	$g(\lambda)$ at the planet surface, m/c^2	Deep of the maximum of $g(\lambda)$, km
The Moon and Terrestrial planets				
Mercury	11.33066	1.153545	3.7	202
Venus	11.07512	1.163695	8.9	404
Earth	12.89487	1.248673	9.8	1039
Moon	4.25882	0.642566	1.6	0
Mars	5.95588	0.849683	3.7	0
The Giant planets				
Jupiter	8.19094	1.984189	24.8	20118
Saturn	5.49708	2.182179	10.4	20796
Uranus	9.24325	2.108185	8.9	7819
Neptune	11.55951	2.085144	11.1	7515

According to the so-called Saigy theorem, the gravity $g(\lambda)$ has a maximum inside the Earth. If the radial model has been used as a basic tool, some additional closed relationships can be derived. Thus, it is necessary to find such a point(s), where the radial derivative $\frac{dg(\lambda)}{d\lambda}$ is equal to zero. As a result, for the stationary point(s) we write first of all the well-known expressions

$$\frac{dg(\lambda)}{d\lambda}=\frac{4\pi G}{3}\left(\rho_m(\lambda)+\lambda\frac{d\rho_m(\lambda)}{d\lambda}\right)=4\pi G\left(\rho(\lambda)-\frac{2}{3}\rho_m(\lambda)\right)=0 \quad \Rightarrow \quad \rho(\lambda)=\frac{2}{3}\rho_m(\lambda). \quad (16)$$

Moreover, since the Roche model is a certain approximation of Gauss's law, instead of numerical integration we use Roche's law, which gives simple estimates in closed form

$$\rho(x)=\rho_0\left(1-Kx^2\right)=a+bx^2,\ (K=\text{const}), \quad (17)$$

where

$$a=\rho_0>0 \quad \text{and} \quad b=\rho_0 K<0, \quad (18)$$

and get immediately

$$\rho_m(\lambda)=a+\frac{3\cdot b}{5}\left(\frac{\lambda}{R}\right)^2=a+\frac{3\cdot b}{5}x^2. \quad (19)$$

Now the solution of (16) leads to the closed expression for the parameter x:

$$x=\frac{\lambda}{R}=\frac{\sqrt{5}\cdot\sqrt{a}}{3\cdot\sqrt{-b}}. \quad (20)$$

This root corresponds to (17) and $a>0$, where the sign of b must be negative: $b<0$. Furthermore applying such dimensionless relationship (20) for $x\in[0,1]$, the following inequality can be obtained

$$\frac{a}{-b}\le\frac{9}{5}, \quad (21)$$

for the coefficients of Roche's model. It should be noted that only a sign of the second radial derivative follows from the coefficient b. For this reason $\frac{d^2g(\lambda)}{d\lambda^2}<0$ at the point (20) and our function $g(\lambda)$ has one maximum only at this point. To calculate the coefficients a,b, we use the following convenient equations

$$a=\frac{5\rho_m(10-21I_m)}{8}, \quad b=\frac{35\rho_m(5I_m-2)}{8}, \quad (22)$$

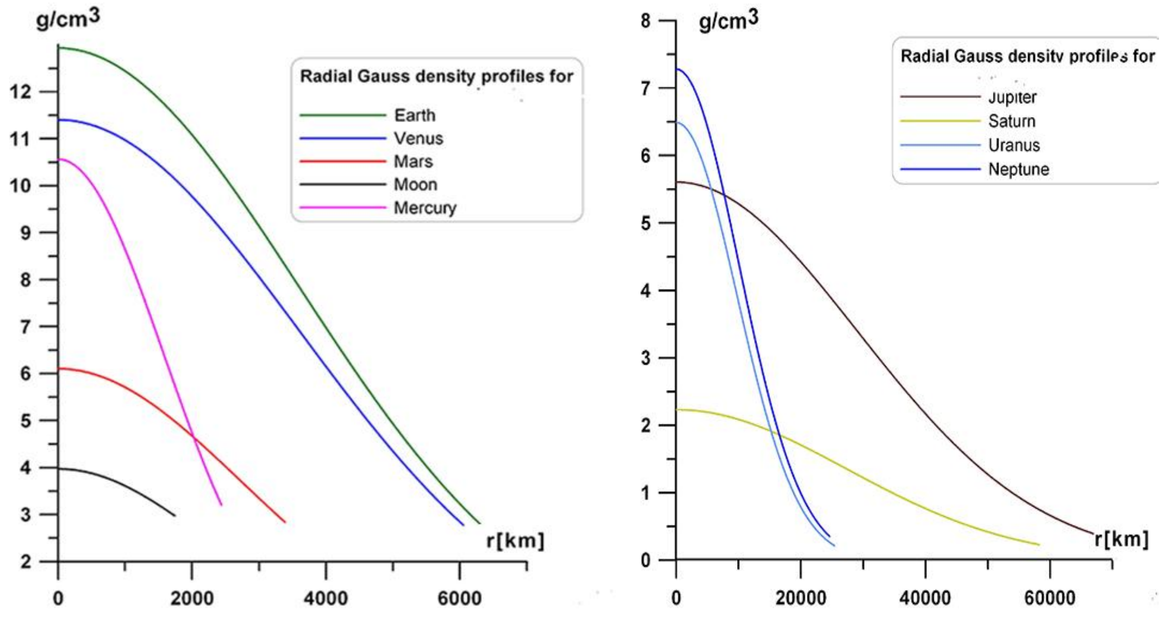
which together with expressions (14) - (15) were used for the calculation of planet's parameters given in Table 2.

It has to be pointed out that in the cases of the Moon and Mars Roche's model cannot allowing positive results for the Seidgy theorem. The formulas (16)–(22) lead to the maximum at the surface of these planets. It is obvious because Roche's law was used for the approval of this theorem. More consistent results can be obtained via the normal distribution.

Planets gravitational potential energy. Conclusion and summery

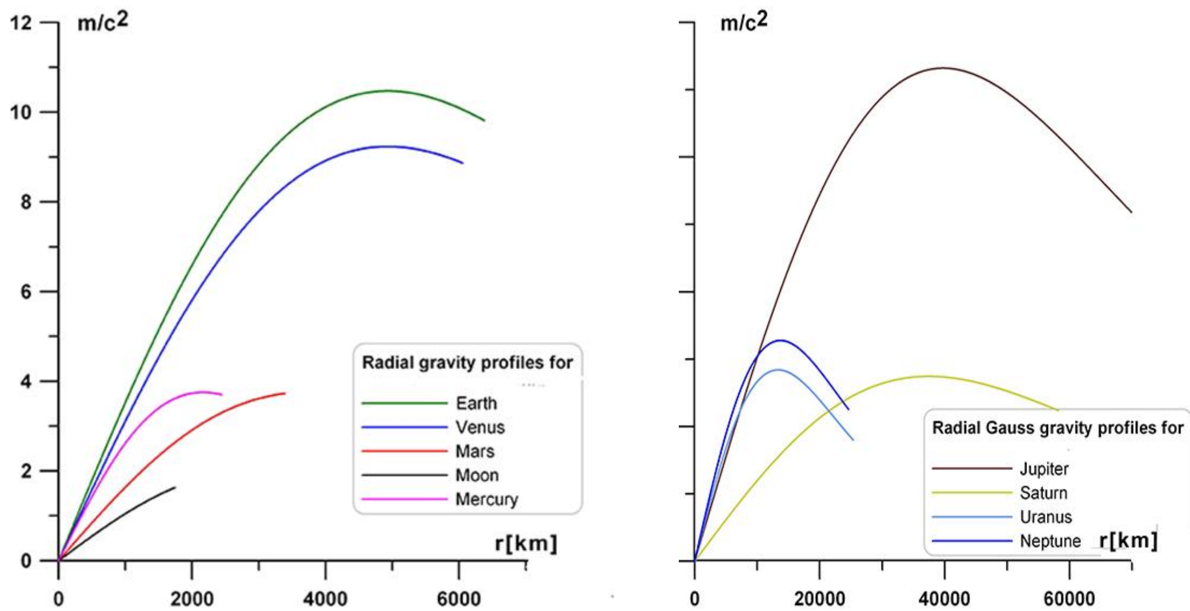
Fig. 2 demonstrates all normal density distribution $\rho(\lambda/R)$ [g/cm^3] for the Moon and terrestrial planets (left) including additionally the outer planets (right). It should be noted that density about the surface of all

Giant planets is close to zero that corresponds to NASA data from Table 1 in the form of inequalities. Fig. 3 reflects the Seigy theorem illustrating gravity distribution $g(\lambda / R)$ [m/c^2] based on the normal density of the Moon and terrestrial planets together with the outer planets. Thus, the normal law of density leads to the most deeper values of maximum as shown in Fig. 3. More detailed analyses allow the same conclusion as shown in Fig. 2 and Fig. 3 together with Table 1 and Table 2.



a *b*

Fig. 2. Normal density distribution $\rho(r)$ [g/cm^3] of the Moon and terrestrial planets (a) including this law for outer planets (b)



a *b*

Fig. 3. Gravity distribution $g(r)$ [m/c^2] based on the normal density of the Moon and terrestrial planets (a) and including this law for the outer planets (b)

One of the unique criteria for the determination of ρ (as the solution of ill-posed problem) is the search for the stationary value of E [Wermer, 1981]. A similar approach follows from Mescheryakov's (1977) result: "If a numerical value of the potential energy E and the density on the Earth's surface σ are given prior, then in the Tikhonov sense the determination of the density ρ transforms into a properly posed problem" [Tikhonov and Arsenin, 1974]. Thus, the work $W = -E$ of the gravitation represents the quadratic functional of ρ [Moritz, 1990] and can be used in the solution of this inverse problem. In view of geophysics the energy E may also be applied as an estimate of a lower limit to the mantle viscosity obtained from the rate of time-changes of E [Rubincam, 1979]. The continuous Legendre–Laplace law, Roche's law, Bullard's model, Gaussian distribution of the Earth and the piecewise continuous Roche profile were applied in Marchenko (2009) for several estimations of E that led to the inequality with minimum limit corresponding to Gauss' continuous profile and very close to the piecewise PREM model [Marchenko, Zayats, 2011]. Therefore, the comparison of all the E-estimates gives

$$E_{\mathrm{Gauss}} \le E_{\mathrm{Earth}} \le E_{\mathrm{H}} < -W_{\min} . \qquad (23)$$

The conventional expression for the Earth's gravitational potential energy $E = -W$ reads [Gauss, 1867; Moritz, 1990]:

$$E = -\frac{1}{2}\int_{\tau} V_i(r,\vartheta,\lambda)\cdot\rho(r,\vartheta,\lambda)\cdot d\tau , \qquad (24)$$

According to (Gauss, 1867; Moritz, 1990) the work $W = -E$ has some minimal value $W_{\min}$ if all masses are concentrated on the level surface σ where the gravitational potential $V_0 = \mathrm{const}$ and the interior is empty. In this case, the internal potential $V_i = V_0 = \mathrm{const}$ represents the harmonic function inside the surface σ and leads to zero Dirichlet's integral. Thus, the minimum amount of $W = -E$ becomes

$$W_{\min} = M\cdot V_0 / 2 , \qquad (25)$$

and represents the solution of the variational Gauss problem [Gauss, 1840, 1867]. Substitution of eq. (25) into Eq. (24) leads to the following fundamental formulae

$$E = -(W_{\min} + \Delta W) . \qquad (26)$$

Thus, ot $\Delta W = \frac{1}{8\pi G}\int_{\tau} D(V_i, V_i)\,d\tau$ her kind of expression for E, given under the assumption that the boundary σ is a level surface, provides a simple estimation of $W_{\min}$ and remarkable interpretation of the deviation ΔW from this minimal amount $W_{\min}$, i.e. in terms of a non-zero Dirichlet's integral when all masses are distributed inside τ according to an adopted density law.

Final expressions based on eq. (26) read

$$-E = W = \frac{GM^2}{2R} = \frac{\pi^2 G\rho_0^2 R^5 \exp(-2\gamma^2)}{2\gamma^6}\left[2\gamma - \sqrt{\pi}\exp(\gamma^2)\,\mathrm{erf}(\gamma)\right]^2 , \qquad (27)$$

$$-\Delta E = \Delta W = \frac{\sqrt{\pi^5}\, G\rho_0^2 R^5}{2\gamma^6}\left[\sqrt{2}\gamma\cdot \mathrm{erf}(\sqrt{2}\gamma) - \sqrt{\pi}\,\mathrm{erf}(\gamma)^2\right], \qquad (28)$$

Table 3 contains estimates of the energy $E = -(W_{\min} + \Delta W)$ separately for each planet, but in all cases, the normal law of density was applied.

Table 3

Results of the calculation of the work of gravitation $W_{\min}$, ΔW and the energy E

Planet	$W_{\min}$	ΔW	$E = -(W_{\min} + \Delta W)$
The Moon and Terrestrial planets			
Mercury	1.4902260×10^{37} erg	4.6793022×10^{36} erg	-1.9581562×10^{37} erg
Venus	1.3063239×10^{39} erg	4.1336440×10^{38} erg	-1.7196883×10^{39} erg
Earth	1.8682711×10^{39} erg	6.3194654×10^{38} erg	-2.5002176×10^{39} erg
Moon	1.0381252×10^{36} erg	2.3922314×10^{36} erg	-1.2773484×10^{36} erg
Mars	4.0513002×10^{37} erg	1.0374989×10^{37} erg	-5.0887992×10^{37} erg
The Giant planets			
Jupiter	1.7190280×10^{43} erg	1.1264831×10^{43} erg	-2.8455112×10^{43} erg
Saturn	1.8500793×10^{42} erg	1.4445069×10^{42} erg	-3.2945862×10^{42} erg
Uranus	9.9237757×10^{40} erg	7.2661139×10^{40} erg	-1.7189889×10^{40} erg
Neptune	1.4213884×10^{41} erg	1.0197873×10^{41} erg	-2.4411757×10^{41} erg

It has to be pointed that the gravitational potential energy in the Table 3 were computed using the direct formula (29) [Marchenko, 2009] based on the expression (24):

$$E = \frac{\pi^2 G \delta_0^2 R^5}{\beta^4}\left(\frac{2\sqrt{\pi}\exp(-\beta^2)\operatorname{erf}(\beta)}{\beta} - 2\exp(-2\beta^2) - \frac{\sqrt{2\pi}\operatorname{erf}(\sqrt{2}\beta)}{2\beta}\right), \qquad (29)$$

without assuming that the surface Σ of each planet is a level surface. Because the stratification of the planets was accepted as spherical, we came to conclusion that formulas (29) and $E = -(W_{\min} + \Delta W)$ give the same results. Another conclusion will connect with the general amount of gravitational potential energy. According to the Encyclopedia of the Solar System (2015), we will recall that such giant planets as Jupiter, Saturn, Uranus, and Neptune have accounted for 99.5 % of all the planetary mass in the solar system. The gravitational potential energy of these outer planets consists of 74.6 % of all planetary energy from Table 3. The amount of energy of all terrestrial planets, including the Moon, has accounted for 25.4 % of all planetary energy.

Mathematically speaking, other types of the expressions (27)–(28) for E give an remarkable interpretation of the deviation of the work of gravitation for the infinite transition of the masses to the planet's surface with a minimum value $W_{\min}$ (corresponds to the first step or (27) and the work of gravitation having total Earth's mass M, from a state of an infinite diffusion before its distribution on the planet's surface Σ (Thomson and Tait, 1883)). Thus, in view of the nonzero Dirichlet's integral $\Delta W > 0$, Table 3 shows that the value of the Dirichlet's integral ΔW for each planet significantly depends on its radius. With a smaller radius, we get a smaller impact, which has a simple explanation. Indeed, the Dirichlet's integral (formula (28) corresponds to the second step or to the work of gravitation to transport masses having the total mass of the Earth M, "from a state of infinite diffusion on the planet's surface to its actual state inside the planet"). That is, the work of gravitation, which has the total mass of the Earth, corresponds to formula (29) and again combines the definition (Thomson and Tait, 1883) and its transition from a state of an infinite diffusion to the actual distribution within the planet. Increasing the radius of the planet leads to a corresponding increase in the work of gravitation. This state of affairs can be changed only by the application of different laws of density, which best approximates its piecewise continuous distribution. However, the normal density distribution used in this work is, in our opinion, one of the best in terms of approximation and transformation into a finite element method for the radial density profile.

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Олександр МАРЧЕНКО, Сергій ПЕРІЙ, Іван ПОКОТИЛО, Зоряна ТАРТАЧИНСЬКА

[1-4]Lviv Polytechnic National University Національний університет "Львівська політехніка",
Інститут геодезії, вул. Карпінського, 6, Львів, 79013, Україна,
e-mail: march@pancha.lviv.ua; periy_ss@ukr.net; ivan.y.pokotylo@lpnu.ua; ztartachynska@yahoo.com,
[1]https://orcid.org/0000-0002-9315-5903, [2]https://orcid.org/0000-0002-2489-3275,
[4]https://orcid.org/0000-0002-5704-6128

ГРАВІТАЦІЙНА ПОТЕНЦІАЛЬНА ЕНЕРГІЯ ТА ОСНОВНІ ПАРАМЕТРИ ЗЕМНИХ ПЛАНЕТ І ПЛАНЕТ-ГІГАНТІВ

Основною метою цього дослідження (перший етап) стало накопичення відповідного набору фундаментальних астрономо-геодезичних параметрів для їх подальшого використання з метою визначення складових розподілів густини для земних та зовнішніх планет Сонячної системи (на інтервалі більше ніж десять років). Початкові дані отримано у результаті кількох кроків загального способу дослідження Сонячної системи із виконанням ітерацій за допомогою різних космічних апаратів та місій. Механічні та геометричні параметри планет дають змогу знайти розв'язання оберненої гравітаційної задачі (другий етап) у разі використання гауссового розподілу густини для Місяця та земних (Меркурій, Венера, Земля, Марс) і планет-гігантів (Юпітер, Сатурн, Уран, Нептун). Цей закон розподілу густини Гаусса (або нормальний розподіл) вибрано як частковий розв'язок рівняння Адамса – Вільямсона та найкраще наближення кусково-радіального профілю Землі, ураховуючи модель PREM на основі незалежних сейсмічних швидкостей. Цей висновок, як гіпотеза вже зроблений для Землі, використано для вирішення проблеми апроксимації для інших планет, щодо яких ми сподіваємося вирішити обернену гравітаційну проблему в разі застосування розподілу густини Гаусса для інших планет, оскільки сейсмічна інформація в такому випадку майже відсутня. Тому, якщо ми можемо знайти стійкий розв'язок для оберненої гравітаційної задачі та відповідний розподіл густини Гаусса, апроксимований із належною якістю, то приходимо у результаті до стабільного визначення гравітаційної потенційної енергії земних та гігантських планет. Крім нормального закону густини планети, визначено гравітаційну потенціальну енергію, інтеграл Діріхле та інші фундаментальні параметри планет Сонячної системи. Це дослідження здійснюється вперше як статичне, щоб уникнути можливих залежностей від часу в гравітаційних полях планет.

Ключові слова: фундаментальні астрономо-геодезичні параметри; розв'язання оберненої гравітаційної задачі; розподіл густини Гаусса; інтеграл Діріхле.

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