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THE GRADIENT CONSTRUCTION APPROACH ANALYSIS OF THE THREE-DIMENSIONAL MASS DISTRIBUTION FUNCTION OF THE ELLIPSOIDAL PLANET

To investigate the features of the algorithm implementation for finding the derivatives of the spatial distribution function of the planet’s masses with the use of high-order Stokes constants and, on the basis of this, to find its analytical expression. According to the given methodology, to carry out calculations with the help of which to carry on the study of dynamic phenomena occurring inside an ellipsoidal planet. The proposed method involves the determination of the derivatives of the mass distribution function by the sum, the coefficients of which are obtained from the system of equations, which is incorrect. In order to solve it, an error-resistant method for calculating unknowns was used. The implementation of the construction is carried out in an iterative way, while for the initial approximation we take the three-dimensional function of the density of the Earth's masses, built according to Stokes constants up to the second order inclusive, by dynamic compression by the one-dimensional density distribution, and we determine the expansion coefficients of the derivatives of the function in the variables x, y, z to the third order inclusive. They are followed by the corresponding density function, which is then taken as the initial one. The process is repeated until the specified order of approximation is reached. To obtain a stable result, we use the Cesaro summation method (method of means). The calculations performed with the help of programs that implement the given algorithm, while the achieved high (ninth) order of obtaining the terms of the sum of calculations. The studies of the convergence of the sum of the series have been carried out, and on this basis, a conclusion has been made about the advisability of using the generalized finding of the sums based on the Cesaro method. The optimal number of contents of the sum terms has been chosen, provides convergence both for the mass distribution function and for its derivatives. Calculations of the deviations of mass distribution from the mean value (“inhomogeneities”) for extreme points of the earth's geoid, which basically show the total compensation along the radius of the Earth, have been performed. For such three-dimensional distributions, calculations were performed and schematic maps were constructed according to the taken into account values of deviations of three-dimensional distributions of the mean (“inhomogeneities”) at different depths reflecting the general structure of the Earth's internal structure. The presented vector diagrams of the horizontal components of the density gradient at characteristic depths (2891 km – core-mantle, 700 km – middle of the mantle, also the upper mantle – 200, 100 km) allow us to draw preliminary conclusions about the global movement of masses. At the same time, a closed loop is observed on the “core-mantle” edge, which is an analogy of a closed electric circuit. For shallower depths, differentiation of vector motions is already taking place, which gives hope for attracting these vector-grams to the study of dynamic motions inside the Earth. In fact, the vertical component (derivative with respect to the z variable) is directed towards the center of mass and confirms the main property of mass distributions – growth when approaching the center of mass. The method of stable solution of incorrect linear systems is applied, by means of which the vector-gram of the gradient of the mass distribution function is constructed. The nature of such schemes provides a tool for possible causes of mass redistribution in the middle of the planet and to identify possible factors of tectonic processes in the middle of the Earth, i.e indirectly confirms the gravitational convection of masses. The proposed technique can be used to create detailed models of density functions and its characteristics (derivatives) of the planet's interior, and the results of numerical experiments – to solve tectonics problems.

Key words: incorrect problem; Fire – Cesaro method; Earth; PREM model; Stokes constants.

Introduction

The determining characteristic of the structure of the Earth internal structure is the function of mass distribution of its bowels [Fys, et al., 2021]. It is an exhaustive factor in the formation of the properties of terrestrial matter and in combination with the figure of the planet forms an external gravitational field, which is characterized by spatial dependence (three-dimensionality). It cannot be the result of the action of real one-dimensional models of mass distribution density used in geophysics. After all, the radial models of the planet's subsoil distribution created using seismology data involve the use of only Stokes constants of zero and second orders (mass, moment of inertia) [Dzewonski, & Anderson, 1981]. Today, the most widely used is the standard model, PREM [Dzewonski, & Anderson, 1981], which generally characterizes the internal structure of the planet, and therefore can be the basis for further constructions of three-dimensional density models using higher-order gravitational field coefficients. In such a situation, the solution of the problem is possible only approximately and has an iterative character [Tserklevych, et al., 2012]. The zero approximation of the value is usually taken as a three-dimensional density model [Meshcheryakov, & Fys, 1986] based on one of the standard models, such as PREM [Dzewonski, & Anderson, 1981] and consistent with Stokes constants up to and including the second order and dynamic compression. The fact of taking into account the coefficients of potential decomposition to a certain order [Meshcheryakov, & Fys, 1986] generates an iterative scheme for constructing such models that reflect in sufficient detail the nature of the distribution of planetary masses [Meshcheryakov, & Fys, 1990].

The dynamics of phenomena occurring in the middle of the Earth involves a change in the distribution of masses in both radius and longitude, i.e the gradient of density change is also a three-dimensional function of the three coordinates. Research in this direction and geophysics has hardly been conducted and relate mainly to changes in the radius distribution function. The possibility of latitudinal changes was only discussed. Therefore, given the efficiency of approximation of the function by the sum of the corresponding polynomials [Meshcheryakov, et al., 1986; Meshcheryakov, et al., 1990], it is natural to apply it not only to the mass distribution function itself, but also to its derivatives. This work is devoted to the attempt of such adaptation. The considered method is also an approximation that can be partially estimated (values on the Earth's surface are used), which makes it possible to analyze the iterative process. The derivative distribution functions constructed using the proposed method are more informative and give a more detailed picture of the distribution of the planet's masses, including their possible movement in the middle of the Earth, and the obtained model function gives a picture of the mass distribution. Maps for the vertical derivative confirm the growth of the mass distribution function with increasing depth in the middle of the planet.

Purpose

To propose and investigate the method of stable construction of derivatives of the three-dimensional mass distribution function of the planet together with the Stokes constants of high orders. Based on it, to study the internal structure of the Earth and possible mass movements in the middle of the Earth.

Methodology

The main theses of the approximate method for the gradient construction and the distribution function of the interior masses of an ellipsoidal planet.

The use of a representation with systems of biorthogonal systems for the derivatives of the mass distribution function of the planet's interior was proposed in [Fys, et al., 2020]. The essence of this method is to represent the image of lumpy continuous derivatives of density functions using orthogonal systems (later biorthogonal $\{W_{mnk}\},\{\omega_{mnk}\}$) polynomials of three variables in an ellipsoid $\tau:\left\{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\le 1\right\}$, namely:

$$\frac{\partial\delta}{\partial x_i}=\frac{1}{a_e}\sum_{t=m+n+k=0}^{N} a^i_{mnk}W_{mnk}(x_1,x_2,x_3)\,, \qquad (1)$$

where

$$a^s_{vnk}=\frac{\int_\tau \frac{\partial\delta}{\partial x_i}\omega_{mnk}\,d\tau}{\int_\tau \omega_{mnk}W_{mnk}\,d\tau}\,, \qquad (2)$$

and is a linear combination of the following values (power moments of the mass distribution function of the planet):

$$I^i_{pqs}=\frac{1}{Ma_1^pa_2^qa_3^s}\int_\tau x_1^px_2^qx_3^s\frac{\partial\delta}{\partial x_i}\,d\tau\,,\ p+q+s=t\,, \qquad (3)$$

where M – planet mass, a_e – equatorial radii of the Earth, a_1,a_2,a_3 – half-axis of the ellipsoid [Fys, et al., 2016,2018].

The quantities (3) (power moments of the derivatives) are determined through the power moments of the density function δ using the Ostrogradsky formula

$$I^i_{pqs}=\frac{1}{Ma_1^pa_2^qa_3^s}\left(-p\int_\tau x_1^{p-1}x_2^qx_3^s\delta\,d\tau+\right.$$

$$\left.+\iint_\sigma x_1^px_2^qx_3^s\delta\left(x_1,x_2,x_3\right)\cos\alpha_i\,d\sigma\right), \qquad (4)$$

where $\left(\cos\alpha_1,\cos\alpha_2,\cos\alpha_3\right)$ – normal vector to the surface, with next components

$$\cos\alpha_i,=\frac{x_i}{D\left(x_1,x_2\right)},\ i=1,2,\ \cos\alpha_3=\frac{1}{D\left(x_1,x_2\right)},$$

$$x_3 = \pm a_3 \sqrt{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}} = f\left(x_1, x_2\right),$$

$$D\left(x_1, x_2\right) = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2}. \qquad (5)$$

The representation of the surface of the ellipsoid in formulas (5) can be considered in another form, for example, expressed x_1 through x_2, x_3. Therefore, the quantities (4) are determined by the power moments of the density function, the order of which is one less than in (3), and the expressions [Fys, et al., 2020]

$$\sigma_{pqs} = \frac{1}{M a_1^p a_2^q a_3^s} \iint_\sigma x_1^p x_2^q x_3^s \delta \cos\alpha_i d\sigma = \frac{1}{M a_1^p a_2^q a_3^s} \iint_\sigma x_1^{p_1} x_2^{q_1} x_3^{s_1} \delta \frac{d\sigma}{D\left(x_2, x_3\right)}, \qquad (6)$$

Power moments are determined from a system of equations derived from an expression for spherical functions [Fys, et al., 2019]:

$$C_{nk} + iS_{nk} = \frac{RR(n-k)!}{2^k} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m \int_\tau x_3^{n-k-2m}\left(x_1^2 + x_2^2\right)^m \left(x_1 + i x_2\right) d\tau}{2^{2m} m!(m+k)!(n-k-2m)!} = \sum_{p+q+s=n} \left(\alpha_{pqs} + i\beta_{pqs}\right) I^{pqs}, \qquad (7)$$

$0 \le k \le n$

The system of equations at $n > 2$ is ambiguous and can be solved by performing additional conditions (for example, under the least squares). The power moments of the density up to and including the second order are determined from these observations, and therefore they can be used to determine the values I^i_{pqs} with formula (4) up to and including the third order. In this case, there is a need to determine the additional values σ_{pqs} included in the transformation of the Stokes constant as follows:

$$\begin{cases} C_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(C^*_{nk} + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs}\right) \\ S_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(S^*_{nk} + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs}\right) \end{cases} \qquad (8)$$

where

$$C^*_{nk} = -\int_\tau \frac{\partial \delta}{\partial x_i} U^i_{nk} d\tau, \quad S^*_{nk} = -\int_\tau \frac{\partial \delta}{\partial x_i} V^i_{nk} d\tau,$$

$U^i_{nk} = \int_0^{x_i} U_{nk} dx_i, V^i_{nk} = \int_0^{x_i} V_{nk} dx$ – antiderivatives of spherical functions (polynomials of n + 1 order by variables x_1, x_2, x_3).

The set of equations (8) is supplemented by identities:

$$\sigma_{p+2,q,s} + \sigma_{h,q+2,s} + \sigma_{p,q,s+2} = \sigma_{pqs},$$

which in symbolic form can be submitted

$$\left(\sigma_{p00} + \sigma_{0q0} + \sigma_{00s}\right)^{2t} = \sigma_{pqs}, \quad p,q,s = 0 \text{ or } 1 \qquad (9)$$

where $\left(\sigma_{p00}\right)^i \left(\sigma_{0q0}\right)^j \left(\sigma_{00s}\right)^l = \sigma_{p+2i,q+2j,s+2l}$, $i + j + l = t$.

The system of equations (8) is divided into four groups with the corresponding number of equations and unknowns (Table 1):

$$\begin{array}{ll} I. & \sigma_{00n}, \sigma_{20n-2}, \sigma_{02n-2}, \sigma_{40n-4}, \sigma_{22n-4}, \sigma_{04n-4}, .., \sigma_{n-T0T}, .., \sigma_{T,n-T,T}, \\ II. & \sigma_{10n-1}, \sigma_{30n-3}, \sigma_{12n-2}, \sigma_{50n-5}, \sigma_{32n-5}, \sigma_{14n-5}, .., \sigma_{n-T0T}, .., \sigma_{Tn-T,T}, \\ III. & \sigma_{01n-1}, \sigma_{21n-3}, \sigma_{03n-3}, \sigma_{41n-5}, \sigma_{23n-5}, \sigma_{05n-5}, .., \sigma_{n-T,T,T}, .., \sigma_{Tn-T,T}, \\ IV. & \sigma_{01N-1}, \sigma_{21N-3}, \sigma_{03N-3}, \sigma_{40N-}, \sigma_{22N-4}, \sigma_{04N-4}, .., \sigma_{N-T0T}, .., \sigma_{TN-T,T}, T = n - 2\left[\frac{n}{2}\right]. \end{array} \qquad (10)$$

Table 1

The number of equations and number of unknowns of each group (10)

Name	Number of equations r	Number unknowns t
I	$3\left(\left[\frac{n}{2}\right]+1\right)$	$\frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+3\right)$
II, III, IV(n – paired)	$3\left(\left[\frac{n-1}{2}\right]+1\right)$	$\frac{1}{2}\left(\left[\frac{n-1}{2}\right]+1\right)\left(\left[\frac{n-1}{2}\right]+3\right)$
IV(n – unpaired)	$\left[\frac{n}{2}\right]$	$\frac{n(n+1)}{2}$

In accordance with this, the systems will write:

$$B_i = A_i X_i, \quad i = 1,2,3,4, \tag{11}$$

where B_i – matrices of dimension $r \times n$, the elements of which are coefficients for powers *x*, *y*, *z* of the corresponding spherical functions.

Analyzing the formulas of Table 1 you can set the number of equations and the rank of each of the systems used to determine the solution.

Table 2

The number of equations and unknowns of each group (10) for different orders

Order	*n* group	Number equation	Number unknouns	Rank	Order	*n* group	Number equation	Number unknouns	Rank
2	I	7	6	6	5	I	10	10	9
	II	3	3	3		II	10	10	9
	III	3	3	3		III	10	10	9
	IV	3	3	3		IV	7	6	6
3	I	7	6	6	6	I	13	15	10
	II	7	6	6		II	10	10	10
	III	7	6	6		III	10	10	10
	IV	1	1	1		IV	10	10	10
4	I	10	10	8	7	I	13	15	12
	II	7	6	6		II	13	15	11
	III	7	6	6		III	13	115	11
	IV	7	6	6		IV	10	10	10
8	I	16	21	11	8	III	13	15	12
	II	13	15	10		IV	13	15	10

As can be seen from table 2, the ambiguity of the solution appears only at *n* = 4 (the first group), which is undoubtedly a positive aspect of the proposed method. For the same case and orders above the fourth system (7) has many solutions. It is common to look for a solution that has the property of minimal deviation from zero (known in the literature as "pseudo-solution"). Such a solution is a generalization of the ordinary, because it coincides with it in the case of unambiguity of the system. One of the known algorithms for finding a generalized solution of a system of equations is the following:

1. Reduce the system to normal

$$\begin{aligned} &B_i^T B_i = B_i^T A_i X_i, \quad i = 1,2,3,4,\\ &A_i' = B_i^T B_i, \quad b_i = B_i^T A_i\\ &A_i' X = b_i, \quad m = n,\\ &\det(A_i') = 0 \quad (\det(A_i') \approx 0),\\ &r = rang(A_i') \le n \end{aligned} \tag{12}$$

2. The solution of system (12) is determined by a sequence of matrix operations [Syavavko, & Rybytska, 2000]

$$s_1 = Sp(A_i') = \sum_{i=1}^{n} a_{ij}', \quad C_0 = E, \quad E \text{ - identity matrix,}$$

$$C_1 = C_0 A_i' + s_1 E, \quad s_2 = \frac{1}{2} Sp(C_1 A_i'),$$

...

$$\begin{aligned} &C_{r-1} = C_{r-2} A_i' + s_{r-1} E, \quad s_{r-1} = \frac{1}{r-1} Sp(C_{r-2} A_i'),\\ &s_r = \frac{1}{r} Sp(C_{r-1} A_i''), \quad X = \frac{1}{s_r} C_{r-1} b_i' \end{aligned} \tag{13}$$

The expression of the subsoil density function of the planet is unknown, then we can determine only the approximate values σ_{pqs} (6). In addition, with increasing order of approximation, the number of unknowns may exceed the number of equations, and therefore the results should be interpreted taking into account the conditions, using Stokes constant higher orders (*t* > 2) is possible only approximately [Meshcheryakov, 1990; Meshcheryakov, et al., 1986].

Let's select the steps of the algorithm.

Without an exact expression of the distribution function δ_2, we choose its three-dimensional approximation, taken from work [Fys, et al., 2020]

$$\delta_2(x_1, x_2, x_3) = \delta^0(\rho) + \sum_{m+n+k=0}^{2} b_{mnk} W_{mnk}(x_1, x_2, x_3) \tag{14}$$

where $\delta^0(\rho)$ – spherically symmetrical radial model, for example, PREM [Dzewonski, & Anderson, 1981].

Decomposition coefficients (12) are determined by formulas:

$$b_{000}=\delta_c\left(1-\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2 d\rho\right)\ b_{110}=35\delta_c S_{21},\ b_{101}=35\delta_c C_{21},\ b_{011}=35\delta_c S_{21},$$

$$b_{200}=\frac{7}{2}\delta_c\left[5\left(\frac{C_{20}}{2H}+2C_{22}\right)-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4 d\rho+\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2 d\rho\right],$$

$$b_{020}=\frac{7}{2}\delta_c\left[5\left(\frac{C_{20}}{2H}-2C_{22}\right)-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4 d\rho+\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2 d\rho\right], \quad (15)$$

$$b_{002}=\frac{7}{2}\delta_c\left[5\left(1-\frac{1}{2H}\right)C_{20}-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4 d\rho+3\int_0^1\delta^0(\rho)\rho^2 d\rho\right].$$

The power moments of the density of the second order are calculated as follows [Meshcheryakov, 1991]:

$$I_{200}=\frac{-C_{20}}{2H}+2C_{22},\ I_{020}=\frac{-C_{20}}{2H}-2C_{22},$$

$$I_{002}=C_{20}\left(1-\frac{1}{2H}\right),\quad I_{101}=C_{21},$$

$$I_{011}=S_{21},\quad I_{110}=\frac{1}{2}S_{22}. \quad (16)$$

For such assumptions, the system (8) under condition (9) for n = 1.2 gives a unique value σ_{pqs}. They can also be calculated for the function δ_2 directly by the formulas:

$$\sigma_{pqs}=\frac{1}{Ma_1^p a_2^q a_3^s}\iint_{D^+_{x_2,x_3}} x_1^{p-1}x_2^q x_3^s\delta_2 dx_2 dx_3+\iint_{D^-_{x_2,x_3}} x_1^{p-1}x_2^q x_3^s\delta_2 dx_2 dx_3\ (p>0),$$

$$\sigma_{pqs}=\frac{1}{Ma_1^p a_2^q a_3^s}\iint_{D^+_{x_1,x_3}} x_1^{p}x_2^{q-1} x_3^s\delta_2 dx_2 dx_3+\iint_{D^-_{x_1,x_3}} x_1^{p}x_2^{q}-1x_3^s\delta_2 dx_2 dx_3\ (q>0), \quad (17)$$

$$\sigma^3_{pqs}=\frac{1}{Ma_1^p a_2^q a_3^s}\iint_{D^+_{x_1,x_2}} x_1^{p}x_2^q x_3^{s-1}\delta_2 dx_2 dx_3+\iint_{D^-_{x_1,x_2}} x_1^{p}x_2^q x_3^{s=1}\delta_2 dx_2 dx_3\ (s>0),$$

Expression for δ_2 on the ellipsoid surface:

$$\delta_2=b_{000}+b_{200}x_1^2+b_{020}x_2^2+b_{002}x_3^2+$$
$$+2\left(b_{110}x_1x_2+b_{101}x_1x_3+b_{011}x_2x_3\right)+ \quad (18)$$
$$+b_{100}x_1+b_{010}x_2+b_{001}x_3$$

this allows us to calculate (17) directly (the coefficient b_{000} coincides with the value of the density in the center of mass [Fys, et al., 2013]).

Therefore, the direct calculation of the values of the surface power moments through the decomposition coefficients after substitution (18) in one of the expressions (17) gives:

$$\sigma_{200}=\frac{\delta_c\left(5b_{000}+\left(3b_{200}+b_{020}+b_{002}\right)\right)}{5},$$

$$\sigma_{020}=\frac{\delta_c\left(5b_{000}+\left(b_{200}+5b_{020}+b_{002}\right)\right)}{5},$$

$$\sigma_{002}=\frac{\delta_c\left(5b_{000}+\left(b_{200}+b_{020}+3b_{002}\right)\right)}{5}.$$

Using identity (9) when $t=1$, we determine σ_{000}, namely:

$$\sigma_{000}=\sigma_{200}+\sigma_{020}+\sigma_{002}$$

The power moments of the first order are defined as follows:

$$\sigma_{100}=b_{100}V,\quad \sigma_{010}=b_{010}V,\quad \sigma_{100}=b_{001}V.$$

Using Stokes constants of the first order set the value $\sigma_{pqs}\left(p+q+s=3\right)$:

$$\sigma_{120}=\sigma_{102}=\frac{b_{100}V}{10},\quad \sigma_{210}=\sigma_{021}=\frac{b_{010}V}{10},$$

$$\sigma_{201}=\sigma_{021}=\frac{b_{001}V}{10},\sigma_{300}=\frac{b_{100}V}{5},$$

$$\sigma_{030}=\frac{b_{010}V}{5},\quad \sigma_{003}=\frac{b_{100}V}{5}.$$

When placing the origin of the coordinate system in the center of mass from the above relations we obtain that $\sigma_{pqs}=0,(p+q+s=3)$ and

$$\sigma_{100}=\sigma_{300}+\sigma_{120}+\sigma_{102}=\frac{2b_{100}V}{5},$$

$$\sigma_{010}=\sigma_{030}+\sigma_{210}+\sigma_{012}=\frac{2b_{010}V}{5},$$

$$\sigma_{001}=\sigma_{003}+\sigma_{201}+\sigma_{021}=\frac{2b_{001}V}{5}.$$

Unpaired values σ_{pqs} of fourth order (p, or q, s – unpaired) are defined by Stokes constants C_{21},S_{22}:

$$\sigma_{400}=V\left(\frac{b_{000}}{5}+\frac{3b_{200}}{35}+\frac{b_{020}}{35}+\frac{b_{002}}{35}\right)$$

$$\sigma_{004}=V\left(\frac{b_{000}}{5}+\frac{3b_{002}}{35}+\frac{b_{020}}{35}+\frac{b_{200}}{35}\right)$$

$$\sigma_{040}=V\left(\frac{b_{000}}{5}+\frac{3b_{020}}{35}+\frac{b_{200}}{35}+\frac{b_{002}}{35}\right)$$

$$\sigma_{220}=V\left(\frac{b_{000}}{5}+\frac{3b_{200}}{35}+\frac{3b_{020}}{35}+\frac{b_{002}}{35}\right)$$

$$\sigma_{202}=V\left(\frac{b_{000}}{5}+\frac{3b_{200}}{35}+\frac{b_{020}}{35}+\frac{3b_{002}}{35}\right)$$

$$\sigma_{022}=V\left(\frac{b_{000}}{5}+\frac{b_{200}}{35}+\frac{3b_{020}}{35}+\frac{3b_{002}}{35}\right)$$

from identity (9) and with the help of these relations, we obtained

$$\sigma_{130}=V\frac{3b_{110}}{40},\quad \sigma_{103}=V\frac{3b_{101}}{40},\quad \sigma_{112}=\sigma_{121}=0,$$

$$\sigma_{031}=V\frac{3b_{011}}{40},\quad \sigma_{013}=\sigma_{013}=V\frac{3b_{011}}{40},\quad \sigma_{121}=\sigma_{112}=0.$$

According to the known values σ_{pqs} and I_{pqs} with formulas (4) we found I^i_{pqs}. As in the work [Fys, et al., 2020] we now determine $\frac{\partial\delta_3}{\partial x_1},\frac{\partial\delta_3}{\partial x_2},\frac{\partial\delta_3}{\partial x_3}$.

With the help of derivatives we establish the form of the approximate expression of the mass distribution function δ_4, which we substitute as the initial one.

2. The power moments of density δ in this case are calculated by the following formula:

$$I^*_{pqs}=\frac{1}{Ma_1^p a_2^q a_3^s}\int_\tau x_1^p x_2^q x_3^s \delta_N d\tau .$$

This approximation makes it possible to simplify the process of calculating the expressions (9) of the system (8), which takes the form:

$$\begin{cases} C_{nk}=\dfrac{1}{M\cdot a_1^p a_2^q a_3^s}\left(C^{**}_{nk}+\dfrac{1}{p_1q_1s_1}\sum\limits_{p+q+s=t}\sigma_{p_2q_2s_2}\alpha_{pqs}\right) \\ S_{nk}=\dfrac{1}{M\cdot a_1^p a_2^q a_3^s}\left(S^{**}_{nk}+\dfrac{1}{p_1q_1s_1}\sum\limits_{p+q+s=t}\sigma_{p_2q_2s_2}\alpha_{pqs}\right) \end{cases}, \quad (19)$$

where

$$C^{**}_{nk}=-\int_\tau\frac{\partial\delta_N}{\partial x_i}U^i_{nk}d\tau+\iint_\sigma \delta_N U^i_{nk}d\sigma=\int_\tau\delta_N U_{nk}d\tau ,$$

$$S^{**}_{nk}=-\int_\tau\frac{\partial\delta_N}{\partial x_i}V^i_{nk}d\tau+\iint_\sigma \delta_N V^i_{nk}d\sigma=\int_\tau\delta_N V_{nk}d\tau .$$

This interpretation justifies minor deviations from zero value $\Delta\sigma_{pqs}$ and provides a basis for the application of method (13) for solving the system of equations (19). In addition, this approach greatly simplifies the calculation of the coefficients of decomposition (1) a^i_{mn}, which don't change at $m + n + k < N$, and is calculated when $m + n + k = N$ by the formula

$$a^i_{mnk}=\sum_{p+q+s\le t} d^i_{pqs}I^i_{pqs}=\sum_{p+q+s\le t} d^i_{pqs}I^{i*}_{pqs}+$$

$$+\sum_{p+q+s=t} d^i_{pqs}\Delta\sigma_{pqs}=\frac{(2t+1)!!}{t!}\Delta\sigma_{pqs}.$$

Given that $\left(a^i_{mnk}\right)^*=\sum\limits_{p+q+s\le t} d^i_{pqs}\left(I^i_{pqs}\right)^*=0.$

According to the specified method we define δ_6.

3. The iterative process is repeated until a predetermined order of approximation N_k.

Results

Research of improvement method and interpretation of numerical experiment

To perform calculations, we use information about the gravitational field from work [Pavlis, et al., 2008), dynamic compression, adopted in astronomy [Yatskiv, 1980], the figure recommended by the Geodetic Union [Moritz, H, 1979] and radial mass distribution or fixed values in the center of the Earth [Fys, et al., 2018]. When holding the Stokes constants obtained up to and including the eighth order, we obtain the model functions of the derivatives by rectangular coordinates and the mass distribution function by them up to the tenth order.

Execution of the program on the basis of the described algorithm allows to execute some conclusions.

Let's analyze the results of calculations. The given method provides some improvements of the resulted algorithm in work [Fys, et al., 2020] and elimination of lacks which are revealed in the course of its realization. The generalizations concern first of all construction of a matrix of connections A_i from (17) and reception of the approximate solution (17) irrespective of conditions of its existence. Comparison of the results of calculations in different ways give similar results, so the application of this algorithm is justified.

Based on this technique, a three-dimensional model of density δ_9 with the involvement of Stokes constants up to and including the eighth order, which retains all the basic properties of the reference model PREM: the magnitude of jumps and the depth of their occurrence, the nature of density change relative radius. In this case, in contrast to the model δ_2, the deviations of the density δ_9 from the average value are more informative, i.e give a more detailed picture of the location of the masses. However, the increase in the order of approximation is accompanied by instability of calculations. Therefore, to obtain stable results, we apply the generalized method of summation by means (Cesare's method [Korn G. A., & Korn T. M., 2000], the essence of which is as follows. For the sum

$$\delta_N = c_0 + c_1 + \ldots + c_N = \sum_{i=0}^{N} c_i$$

defined as the limit of the sequence

$$\delta_l = \sum_{i=0}^{l} \left(1 - \frac{i}{l+1}\right) c_i, \; s = \lim_{l \to \infty} \delta_l$$

For two characteristic points (Indian minimum – No. 2, UK – No. 3), the density function anomalies (deviations from the mean) were calculated, which are determined as follows:

$$\Delta\delta_N = \delta_N(x, y, z) - \delta_N(\rho),$$

where $\delta_N(\rho)$ - averaged over the "unit" sphere, the values of the three-dimensional mass distribution function. For them, graphs are given for various orders of the summation (Fig. 1).

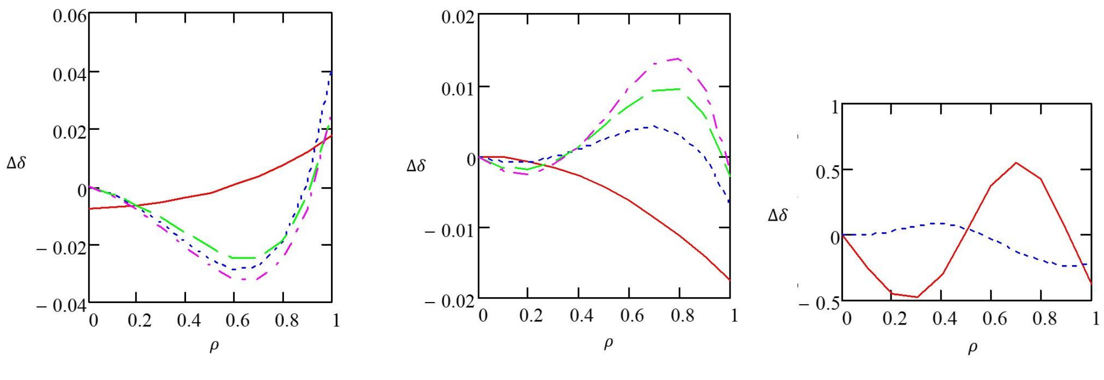
As can be seen from the illustrations, a clear refinement is presented with an increase in the summation order to eight. And already further increasing of summation number leads to distorted results, which is reflected in the graph *c*). This may be due to the fact that no conditions were imposed on the density function (functions of derivatives were created). Therefore, in further studies, we restrict ourselves to the number 8. In support of this assumption, let us also construct graphs of dependences for one of the derivatives.

For characteristic points from Table 3 graphs of distribution of anomalies along radius were constructed (Fig. 3).

Table 3

Geodetic coordinates of the main characteristic points of the Earth's geoid [Meshcheryakov, 1991]

Point number	Geoid heights	Latitude°	Longitude °	Geographical location
1	78	–3	145	Near New Guinea
2	–110	5	79	In the Indian Ocean
3	70	50	–10	Near the UK
4	30	30	–70	Near the Bahamas
5	56	–55	50	Near Antarctica
6	–56	20	–120	In the eastern Pacific



a b *c*

Fig. 1. Graphs of dependence of anomalies on the order of retention of the sum for two characteristic directions

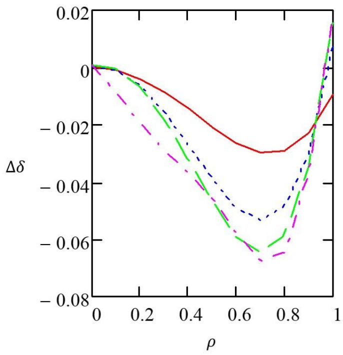


Fig. 2. Graph of the dependence along the radius of the derivative $\frac{\partial \delta_l}{\partial x_1}$ on the order of the sum for point No.1 from Table 3

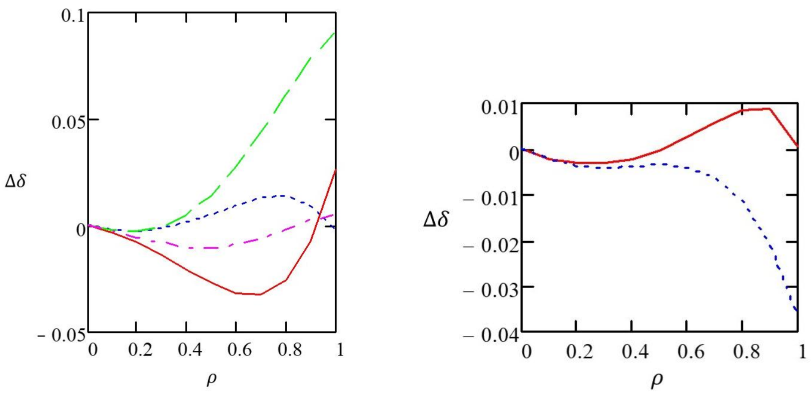


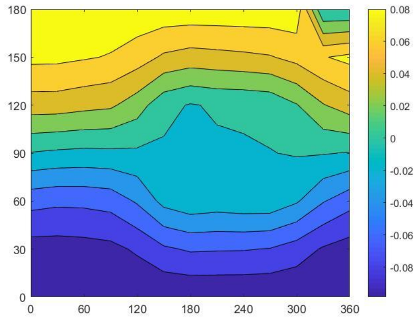
Fig. 3. Graphs of density anomalies for the order of retention of the sum of the number 8 for the characteristic directions of Table 3

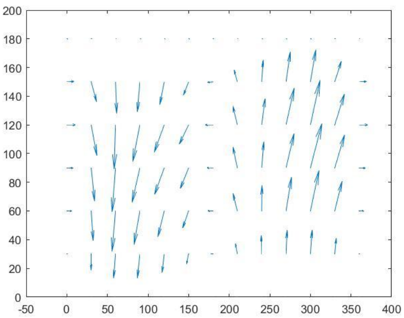
As can be seen from these figures, along the radius of each point, the negative masses at one depth are compensated by positive anomalies at other depths. The graph of the function for point No. 4 falls out of this scheme, for which such a regularity is not observed.

Thus, there is a redistribution of masses at different depths. To do this, maps are constructed that reflect the derivatives of the density function (essentially it is gradient) on different layered surfaces. By the nature of the behavior we can draw an important conclusion about the decline of the function (Fig. 4, a; 5, a; 6 a), density with increasing radius ($\frac{\partial \delta_l}{\partial x_3}$, and in fact $\frac{d\delta}{dr}$), which confirms one of the basic postulates of geophysics. Density gradient components $\frac{\partial \delta}{\partial x_1}, \frac{\partial \delta}{\partial x_2}$ characterize horizontal displacements. They are best depicted in the form of vector-grams, which reflect the directions of horizontal displacements, which is clearly illustrated by the map in Fig. 4, b; 5, b; 6, b (depth 5150 km, inner-outer core boundary) and in Fig. 3 (depth 2891 km). However, the obtained maps show the property of mass movement towards the surface, the cause of which is the rotational motion of the planet. It is characteristic that such clusters are relevant to the Earth entire radius in Fig. 2. At a depth of 5150 km (inner-outer core boundary) and Fig. 3 (depth 2891 km, core-mantle boundary) movement is carried out from the north to the south pole and form a closed circuit that mimics the electric field and can be related to the theory of magnetism. It is characteristic that in higher shells such a phenomenon is absent, for example, on a map of depth 700 km (Fig. 5, b) the vector diagram loses its global character and there are areas associated with compaction and rarefaction of masses. From this it can be argued that this technique has the right to involve it in the study of the internal structure of the Earth. The global nature of the density and therefore cannot be used to the full to explore the Earth. Their low informativeness is explained by the low order of the used Stokes constants and the proposed method (study of derivatives).

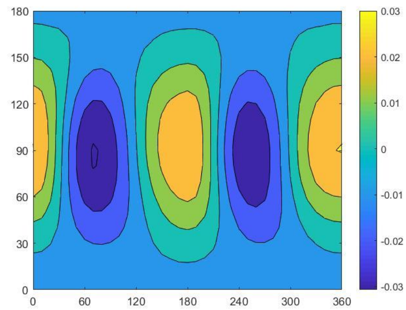
Similarly, we will perform calculations and make appropriate illustrations for a depth of 200 km (Figs. 6–7). The choice of this value is due to the location of the objects of study in the middle of the mantle and their possible impact on the geodynamic processes of the Earth.

Note the most significant points of the results of Figs. 4–7 reflect the picture of the distribution of anomalies of density derivatives along the axis Ox_3 (in a sense – the vertical derivative). It is clear from the figures that the gradient of density anomalies is directed towards the center of mass, because for two depths (Figs. 4–7) for $0 \le \vartheta \le \frac{\pi}{2}$ the angle between it and the axis Ox_3 is obtuse, and for the gap $\frac{\pi}{2} \le \vartheta \le \pi$ it is acute. The nodal point of Fig. 2 is interesting with approximate coordinates ϑ = 120 °, λ = 120 ° which can be interpreted as a point of compression and tension in different directions. Interestingly, it falls into the area of interaction of the Arabian and African tectonic plates. Obviously, a more detailed interpretation requires other ways of comprehensive presentation of information, such as an illustration of the total action of derivatives of variables x_1, x_2.



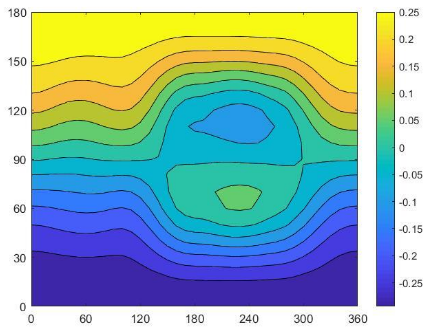


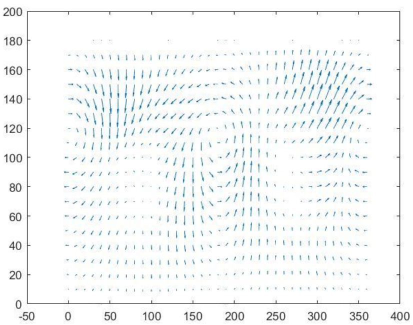
a *b*



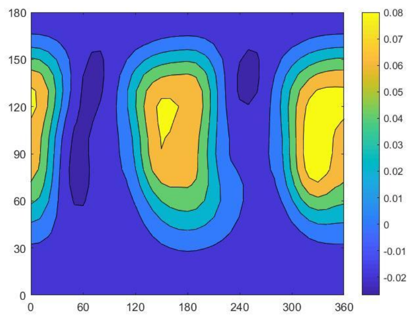
c

Fig. 4. Map of isolines of the derivative with respect to x_3 (a), vector representation of derivatives x_2, x_3 (b) anomalies of the density of the Earth's interior at a depth of 2891 km (c). Isolines sketh through 1.8×10^{-2} g/cm^4 (a) and 6×10^{-3} g/cm^3 (c)



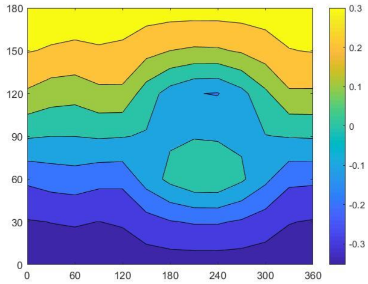


a *b*

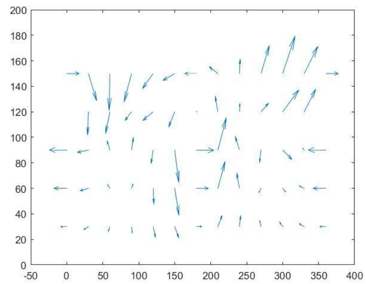


c

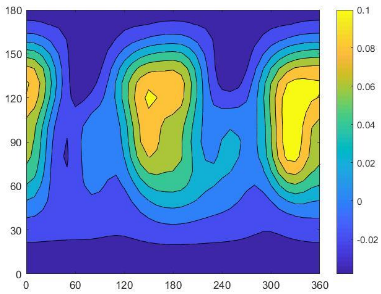
Fig. 5. Map of isolines of the derivative with respect to x_3 (a), vector representation of derivatives x_2, x_3 (b) anomalies of the density of the Earth's interior at a depth of 700 km (c). Isolines sketh through 5×10^{-2} g/cm^4 (a) and 1×10^{-2} g/cm^3 (c)



a

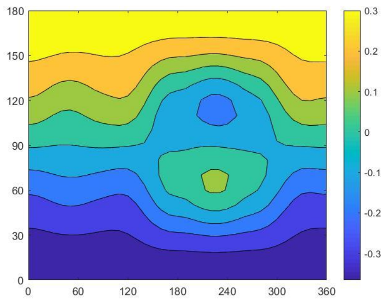


b

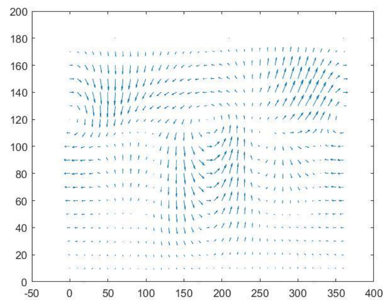


c

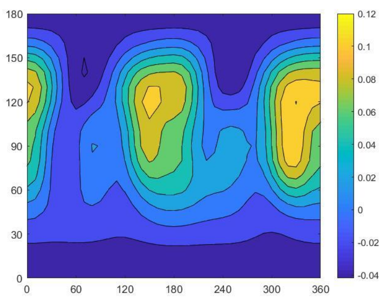
Fig. 6. Map of isolines of the derivative with respect to x_3 (a), vector representation of derivatives x_2, x_3 (b) anomalies of the density of the Earth's interior at a depth of 200 km (c). Isolines sketh through 1×10^{-1} g/cm^4 (a) and 1.7×10^{-2} g/cm^3 (c)



a



b



c

Fig. 7. Map of isolines of the derivative with respect to x_3 (a), vector representation of derivatives x_2, x_3 (b) anomalies of the density of the Earth's interior at a depth of 100 km (c). Isolines sketh through 7×10^{-1} g/cm^3 (a) and 1.6×10^{-1} g/cm^3 (c)

An attempt at such an approach is made in Fig. 4–7. Even the first step in this approach reveals features, namely, the redistribution of masses at the boundary “core-mantle” is carried out from the south to the north pole, which coincides with the action of the magnetic field. Characteristically, in the second case, the picture of movement is completely different, and it can be associated with tectonic movements.

Conclusions

1. The method of approximate construction of the derivatives of the Earth mass distribution function for high orders of the expansion coefficients of the planet's gravitational field has been improved.

2. A stable solution is obtained for finding the expansion coefficients in a series of derivatives and the function itself using the Cesaro method.

3. Vector-schemes built on the “core-mantle” edge confirm the hypothesis about the nature of the magnetic field.

4. For further detailing, further improvement of the algorithm is expected in order to optimize the use of RAM.

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ДОСЛІДЖЕННЯ ОДНОГО МЕТОДУ ПОБУДОВИ ГРАДІЄНТА ТРИВИМІРНОЇ ФУНКЦІЇ РОЗПОДІЛУ МАС НАДР ЕЛІПСОЇДАЛЬНОЇ ПЛАНЕТИ

Мета роботи – дослідити особливості реалізації алгоритму знаходження похідних просторової функції розподілу мас планети із залученням стоксових сталих високих порядків та на основі цього знайти її аналітичний вираз; за наведеною методикою виконати обчислення, за допомогою яких вивчити динамічні явища, що відбуваються всередині еліпсоїдальної планети. Запропонований метод передбачає визначення похідних функції розподілу мас сумою, коефіцієнти якої отримують із системи рівнянь, що є некоректною. Для її розв'язування використано стійкий до похибок метод обчислення невідомих. Побудову реалізовано ітераційним способом, а за початкове наближення взято тривимірну функцію густини мас Землі, побудовану за стоксовими сталими до другого порядку включно, із динамічним стисненням одновимірним розподілом густини. Визначено коефіцієнти розкладу похідних функції за змінними x, y, z до третього порядку включно. Згідно із ними встановлено відповідну функцію густини, яку надалі взято за початкову. Процес повторювали до досягнення заданого порядку апроксимації. Для отримання стійкого результату використано метод підсумування Чезаро (метод середніх). Виконано розрахунки за допомогою програм, що реалізують наведений алгоритм, й досягнуто високий (дев'ятий) порядок отримання членів суми обчислень. Виконано дослідження збіжності суми ряду та на цій основі зроблено висновок про доцільність використання узагальненого знаходження сум на основі методу Чезаро. Вибрано оптимальну кількість утримання членів суми, що забезпечує збіжність як для функції розподілу мас, так і для її похідних. Виконано обчислення відхилень розподілу мас від середнього значення ("неоднорідностей") для екстремальних точок земного геоїда, які загалом свідчать про сумарну компенсацію вздовж радіуса Землі. Для таких тривимірних розподілів виконано обчислення та побудовано картосхеми за врахованими значеннями відхилень тривимірних розподілів від середнього ("неоднорідностей") на різних глибинах, які відображають загальну структуру внутрішньої будови Землі. Наведені вектор-схеми горизонтальних компонент градієнта густини на характерних глибинах (2891 км – ядро–мантія, 700 км – середина мантії, також верхня мантія – 200, 100 км) дають підстави зробити попередні висновки про глобальні переміщення мас. На межі "ядро–мантія" спостерігається замкнений контур, що є аналогією замкненого електричного кола. Для менших глибин вже відбувається диференціація векторних рухів, що дає підстави для залучення цих векторграм до дослідження динамічних рухів всередині Землі. По суті, вертикальна компонента (похідна за змінною z) спрямована до центра мас та підтверджує основну властивість розподілів мас – зростання із наближенням до центра мас. Застосовано методику стійкого розв'язування некоректних лінійних систем, за допомогою якої побудовано векторграми градієнта функції розподілу мас. Характер таких схем дає інструмент для виявлення можливих причин перерозподілу мас всередині планети та можливих чинників тектонічних процесів усередині Землі, тобто опосередковано підтверджується гравітаційна конвекція мас. Запропоновану методику можна використовувати для створення детальних моделей функцій густини та її характеристик (похідних) мас надр планети, а результати числових експериментів – для розв'язання задач тектоніки.

Ключові слова: некоректна задача; метод Фаєра-Чезаро; Земля; модель PREM; стоксові сталі.

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