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The path integral method in interest rate models

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An application of path integral method to Merton and Vasicek stochastic models of interest rate is considered. Two approaches to a path integral construction are shown. The first approach consists in using Wieners measure with the following substitution of solutions of stochastic equations into the models. The second approach is realised by using transformation from Wieners measure to the integral measure related to the stochastic variables of Merton and Vasicek equations. The introduction of boundary conditions is considered in the second approach in order to remove incorrect time asymptotes from the classic Merton and Vasicek models of interest rates. By the example of Merton model with zero drift, a Dirichlet boundary condition is considered. A path integral representation of term structure of interest rate is obtained. The estimate of the obtained path integrals is performed, where it is shown that the time asymptote is limited.

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1. Introduction

Stochastic equations are used in modeling of pricing of various financial instruments: assets, securities, derivatives, and other financial indicators [1, 2]. Stochastic model of asset pricing that is based on Brownian motion, for the first time has been used by Bachelier [1]. Based on it, Bachelier constructed a formula of option price. The main drawback of Bachelier model is that the option price can reach negative values, which does not corresponds to the economic sense. This drawback is absent in a model of geometric (economic) Brownian motion, starting with which Black and Scholes discovered the well-known formula of the option price [1, 2]. The further development of stochastic models is concerned with their application to various problems in financial engineering. For example, description of dynamics of stochastic volatility, modeling of interest rates, the description of term structure of bonds yield, etc. In particular, Merton and Vasicek models of term structure of interest rates contain the same drawbacks as the Bachelier model, mainly that a stochastic variable can have negative values, which do not agree with a sense of the financial indicator.

There are also models that do not contain incorrect solutions. In particular, in Cox-Ingersoll-Ross model, when comparing to the Vasicek model [1,3], the volatility contains a multiplier $\sqrt{r}$. It is obvious then that solutions exist only for positive values $r > 0$. In numbers of models, the multipliers $r^{\nu}, \nu > 0$ are introduced.

Limitations on the domain of acceptable values can be also achieved by introducing boundary conditions [4–6]. In particular, in work [6] a formula of option price was obtained based on the Bachelier model with a boundary condition of a zero probability flow at the boundary $S = 0$ (S is asset price). With the help of boundary conditions in Merton and Vasicek models, one can also restrict the domain of stochastic variable.

A sufficiently effective method of solving such problems is the path integral method [7–9]. Application of path integral method to financial modeling problems is based on known analogies with quantum mechanics and statistical physics. The Schrödinger equation propagator and transition probability of

Fokker–Planck equation are given in a form of the path integral (the Feynman–Kac formula) [10–12]. In the second approach, the path integral for a given stochastic equation is built on the bases of Wiener measure by means of variable substitution [8,13].

In this work, the Merton and Vasicek models are considered using the second approach. Obtained results for term structure of interest rates are found with a different approach [1]. A way of introduction of boundary conditions in the aforementioned approach is analyzed. By example of Merton model, the Dirichlet boundary condition is considered and an expression for term structure of interest rates is obtained.

2. Construction of path integral

It is known [1,2] that discounting of pricing asset $V(t)$ at the time moment $t > t_0$ (in the model of continuous interest rate) satisfies the formula

$$V_0 = V(t)\, e^{-r(t-t_0)}, \tag{1}$$

where $r > 0$ defines a constant interest rate. Usually interest rate is not constant, its dynamics is modeled by the following stochastic equation

$$dr = \mu(r)\, dt + \sigma(r)\, dw, \tag{2}$$

where dw denotes a Wiener's process. The stochastic variable dw is distributed according to the normal law with the parameters $\langle dw\rangle = 0$, $\langle dw^2\rangle = dt$. Variables $\mu(r)$, $\sigma(r)$ define a drift and volatility of the process, which in general also depend on stochastic variable.

Therefore, in the expression (1) one needs to average over implementations of stochastic process

$$V_0 = V(t)\left\langle \exp\left(-\int_{t_0}^{t} r(\tau)\, d\tau\right)\right\rangle_w, \tag{3}$$

where $\langle\ldots\rangle_w$ denotes a mentioned average. Discounting multiplier in (3) defines a term structure of interest rate $r(t)$ [1,2]:

$$P(t,t_0) = \left\langle \exp\left(-\int_{t_0}^{t} r(\tau)\, d\tau\right)\right\rangle_w, \qquad r(t) = -\frac{\ln P(t,t_0)}{t-t_0}. \tag{4}$$

Modeling of dynamics $r(t)$ is one of the most important problems in financial engineering. The meaning of average in (3) becomes clear if to consider breaking the time interval $[t_0, t]$ into smaller sub-intervals with the help of n points t_i $(i = 1,\ldots,n)$. Then the average in (3) is defined by using formula

$$\big\langle(\ldots)\big\rangle_w \approx \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}\sum_{i=1}^{n+1}\frac{\Delta w_i^2}{\Delta t_i}\Big)(\ldots)\prod_{i=1}^{n+1}\frac{dw_i}{\sqrt{2\pi\Delta t_i}}, \tag{5}$$

with the following notations $\Delta w_i = w_i - w_{i-1}$, $\Delta t_i = t_i - t_{i-1}$ $(t_{n+1} = t)$. Integrating in (5) is done based on the known Wiener measure [8,13]. In the limit $\max(\Delta t_i) \to 0$ $(i = 1,\ldots,n)$, $n \to \infty$ for discounting multiplier in (3), a well-known path integral is based on Wiener measure

$$P(t,t_0) = \int_{w_0} \mathcal{D}w(\tau) \exp\left(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w(\tau)}{d\tau}\Big)^2 d\tau\right)\exp\Big(-\int_{t_0}^{t} r(\tau)\, d\tau\Big), \tag{6}$$

$$Dw(\tau) = \prod_{\tau=t_0}^{t}\frac{dw(\tau)}{\sqrt{2\pi d\tau}}. \tag{7}$$

Path integral is also considered as integral over trajectories, where integration is done over all trajectories that start at the point (t_0, w_0) and at the time moment $t > t_0$ pass through all possible points (t, w), $w \in \mathbb{R}$.

Let us build path integrals by example of Merton and Vasicek models, which are one of the simplest and most used models of the term structure of interest rates:

$$dr = \mu\, dt + \sigma\, dw, \tag{8}$$

$$dr = \beta(\mu - r)\, dt + \sigma\, dw, \tag{9}$$

where (8) is Merton model, and (9) is Vasicek model.

According to the formula (6) of calculation of the average value, it is necessary to build solutions of stochastic equations (8) and (9). For the Merton model, the solutions are the following

$$r(\tau) = r_0 + \mu(\tau - t_0) + \sigma\big(w(\tau) - w_0\big), \quad \tau \in [t_0, t]. \tag{10}$$

For Vasicek model, the solution equals [2]

$$r(\tau) = r_0\, e^{-\beta(\tau - t_0)} + \mu\left(1 - e^{-\beta(\tau - t_0)}\right) + \sigma \int_{t_0}^{\tau} e^{-\beta(\tau - \tau')}\, dw(\tau'), \quad \tau \in [t_0, t]. \tag{11}$$

The value w_0 in the expression (10) denotes the stochastic variable $w(t_0)$ at the time t_0.

By substituting solutions (10), (11) into the expression (6) and calculating the path integrals, we obtain the expression for $P(t, t_0)$ and for the term structure of interest rate (4). Calculations themselves are given in Appendix A.

As we have already noticed, in Merton and Vasicek models the term structure of interest rates shows incorrect time dependency [1] (formulae (52) and (57)), which results in negative values of interest rate. It is obvious that this is a result of negative values of $r(\tau)$, as possible solutions of (10), (11). For positive values of $r(\tau) > 0$, the average value is less than 1

$$\left\langle \exp\left(-\int_{t_0}^{t} r(\tau) d\tau\right)\right\rangle_w < 1.$$

In order to satisfy the condition $r(\tau) > 0$, $\tau \in [t_0, t]$ in the integral (6) one should integrate over domain where solutions (10), (11) have positive values. It is easier to form the condition above if the path integral (6) in $r(\tau)$ variables is expressed. For this, let us perform a variable substitution in the integral based on Wiener measure (6) for each of the models are based on equations (10) and (16). In the Merton model, the integral measure receives the following multiplier

$$\prod_{\tau = t_0}^{t} \frac{1}{\sigma}. \tag{12}$$

Next, let us determine dw from the equation (8) and substitute it into the expression for the measure. As a result, we obtain the following path integral for the average in Merton model

$$P(r, r_0, t - t_0) = \int_{r_0}^{r} \mathcal{D}r \exp\left(-\frac{1}{2\sigma^2}\int_{t_0}^{t}\left(\frac{dr}{d\tau} - \mu\right)^2 d\tau\right) \exp\left(-\int_{t_0}^{t} r(\tau)\, d\tau\right), \tag{13}$$

where the following is denoted

$$\mathcal{D}r = \prod_{\tau = t_0}^{t} \frac{dr(\tau)}{\sqrt{2\pi d\tau}}. \tag{14}$$

We obtain the discount multiplier (6) by integrating over r

$$P(r_0, t - t_0) = \int_{-\infty}^{\infty} P(r, r_0, t - t_0)\, dr. \tag{15}$$

In the formula (15), we explicitly separated dependency $P(r_0, t-t_0)$ from initial interest rate value r_0 and that the value depends only on the time difference $t-t_0$. In order to keep the symmetry, which is used in path integrals in physics problems, we integrate over the final point r in (15) separately. The integration done in this way is based on the measure that corresponds to the Merton equation (8), over all trajectories which join the points (t_0, r_0) and (t, r). The path integral (13) is calculated with the closed form [8, 9] and after its substitution into (15), we obtain the same expressions that are shown in Appendix A.

It is more complicated to do a variable substitution in the path integral in Vasicek model. Let us transform the solution (11). For this, in the last term do partial integration and write it in the following form:

$$\begin{gathered} r(\tau) = f_0 + \sigma w(\tau) - \sigma\beta \int_{t_0}^{\tau} e^{-\beta(\tau-\tau')} w(\tau')\, d\tau', \\ f_0 = \mu + (r_0 - \mu - \sigma w_0)\, e^{-\beta(\tau - t_0)}. \end{gathered} \tag{16}$$

We shall consider the solution (16) as a linear integral transformation of Volterra type, which expresses the variables $r(\tau)$ through $w(\tau)$. After a vast transformation of $w(\tau) \to w'(\tau)/\sigma$ in the integral, we will have the same multiplier (12) extracted. The second part of Jacobian transform to the variables $r(\tau)$ is determined by the kernel of integral transformation (16) [8]

$$K(\tau - \tau') = -\beta\, e^{-\beta(\tau-\tau')}. \tag{17}$$

Using a well-known formula of variable substitution in path integral [8] for the mentioned component of Jacobian, we obtain the following

$$\exp\left(-\frac{1}{2}\int_{t_0}^{t} K(0) d\tau\right) = \exp\left(\frac{1}{2}\beta(t - t_0)\right). \tag{18}$$

By substituting (12), (18) and the value of dw from equation (9) into (6), we obtain the following path integral for Vasicek model

$$\begin{aligned} P(r, r_0, t - t_0) = {} & \exp\left(\frac{1}{2}\beta(t-t_0)\right) \\ & \times \int_{r_0}^{r} \mathcal{D}r \exp\left(-\frac{1}{2\sigma^2}\int_{t_0}^{t}\left(\frac{dr}{d\tau} - \beta(\mu - r(\tau))\right)^2 d\tau\right) \exp\left(-\int_{t_0}^{t} r(\tau)\, d\tau\right). \end{aligned} \tag{19}$$

In order to obtain an average, one should integrate the equation (19) over r, by analogy with (15). It is easy to see that the found one, which is based on (15) and (19) term structure of interest rate, coincides with the one found by using the first approach (56). Let us notice that path integrals (13), (19) without exponential multiplier $\exp(-\int_{t_0}^{t} r(\tau)\, d\tau)$ determine transition probability in Merton and Vasicek models (corresponding formulae are given in Appendix B). The transition probability in Vasicek model in the limit $\beta \to 0$, $\beta\mu \to \mu'$ approaches the transition probability of Merton model, which also results from equations (8) and (9). Transition probabilities (60) and (62) satisfy Fokker–Planck equation [8, 14] for stochastic equations (8) and (9) and have a subgroup property

$$K(r, r_0, t - t_0) = \int_{-\infty}^{\infty} K(r, r', t - t')\, K(r', r_0, t' - t_0)\, dr'. \tag{20}$$

Property (20) underlies path integral construction in problems of quantum mechanics and statistical mechanics [7, 8]. Taking into account the property (20), let us write the average value in formulae (52) and (19) in a form of multiple integral

$$P(r, r_0, t - t_0) \approx \int_{-\infty}^{\infty} \prod_{i=1}^{N+1} K(r_i, r_{i-1}, \Delta t_i) \exp\left(-\sum_{i=1}^{n} r_i \Delta t_i\right) \prod_{i=1}^{N} dr_i. \tag{21}$$

Here, the same division is used as in the formula (5) and denoted $\Delta t_i = t_i - t_{i-1}$ and also $t_{N+1} = t$, $r_{N+1} = r$, while $K(r_i, r_{i-1}, t_i - t_{i-1})$ is the transition probability of a respective model in the interval $t \in (t_{i-1}, t_i)$, $i = 1 \ldots N+1$. Path integrals (13), (19) are obtained by passing to the limit in the formula (21).

Based on these considerations, the following approach of introduction of boundary conditions in the considered models arises. For this, in formula (21) we use transition probability that corresponds to certain boundary conditions. As it is known for transition probability, various initial conditions are considered that correspond to the character of the problem [14]. These are the Dirichlet, Neumann, zero probability flow conditions, and others. Let us notice that a similar approach of accounting for boundary conditions has been used in option pricing models [4]. However, formulas which determine option price contain the average only at the final point of time interval while in this case averaging is performed for each interval element.

As a rule, expressions for transition probability with boundary condition contain a set of terms that one fails to transform into a single exponent term. This complicates the expression (21) in a form of path integral, which contains an exponent dependent on some functional of integration variables. Let us illustrate this by the example of the Merton model with additional Dirichlet condition at the boundary $r = 0$ in the case of a zero drift ($\mu = 0$).

3. Merton model with a boundary Dirichlet condition

As it is known the transition probability in Merton model (60) ($\mu = 0$) with Dirichlet boundary condition at the point $r = 0$ equals [8]

$$K_D(r, r_0, t - t_0) = K(r, r_0, t - t_0) - K(-r, r_0, t - t_0). \tag{22}$$

Subgroup property (20) for $K_D(r, r_0, t - t_0)$ is the following

$$K_D(r, r_0, t - t_0) = \int_0^\infty K_D(r, r', t - t')\, K_D(r', r_0, t - t_0)\, dr'. \tag{23}$$

The choice of Dirichlet condition makes sense because the probability of receiving a zero value at $r = 0$ equals zero, hence getting negative values $r < 0$ is impossible.

For further calculations, one needs to substitute the transition probability (22) into the formula (21) for each split interval

$$P(r, r_0, t - t_0) \approx \int_0^\infty \prod_{i=1}^{n+1} K_D(r_i, r_{i-1}, \Delta t_i)\, \exp\left(-\sum_{i=1}^{n} r_i \Delta t_i\right) \prod_{i=1}^{n} dr_i. \tag{24}$$

Integrations over r_i, $i \in \{1, \ldots, n\}$ in (24) are done on a half-plane according to the property (23). To be able to obtain a precise expression in (24), one should perform the limit $\Delta t_i \to 0$, $i \in \{1, \ldots, n\}$, $n \to \infty$.

The main complexity of calculations is related to the multipliers $K_D(r_i, r_{i-1}$ in (24) which are a sum of two terms (22). In order to represent formulas in a closed form, let us perform a number of transformations. With this purpose, let us use a spectral representation of transition probability (60) ($\mu = 0$) [7]

$$K(r, r_0, t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2 k^2 (t - t_0)}\, e^{ik(r - r_0)}\, dk. \tag{25}$$

Respectively, for transition probability with Dirichlet boundary condition (22) let us write the following spectral series

$$K_D(r, r_0, t - t_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2 k^2 (t - t_0)} \sin(ikr)\, \sin(ikr_0)\, dk. \tag{26}$$

After substituting equations for $K_D(r_i, r_{i-1}, \Delta t_i)$, $i \in \{1 \ldots n+1\}$ in spectral form into formula (24) (26) we can integrate over all r_i, $i \in 1, \ldots, n$. As a result, after a number of transformations and simplifications we obtain the following expression for finite approximation (24)

$$P(r,r_0,t-t_0) \approx e^{-\Delta_{n+1}r}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}\sum_{i=1}^{n+1}\Delta_i k_i^2}\prod_{i=1}^{n}\frac{\Delta_i}{(k_{i+1}-k_i)^2+\Delta_i^2}\sin(k_{n+1}r)\sin(k_1 r_0)\prod_{i=1}^{n+1}\frac{dk_i}{\pi}. \quad (27)$$

The expression (27) contains the $n+1$-fold integral over introduced variables k_i, $i \in 1,\ldots,n+1$, and also respective multipliers. Notations $\Delta_i = t_i - t_{i-1}$ where introduced in order to shorten the representation in (27).

Let us note that multipliers in the expression (27) are interconnected. Thus, we perform variable substitution in multiple integral in order to receive product of independent multipliers. It is achieved with the help of variable substitution

$$k_i = k_{n+1} + \sum_{j=i}^{n}\tilde{k}_i, \quad i \in \{1,\ldots,n\}. \quad (28)$$

Here, we consider the variable k_{n+1} as a parameter. It is obvious that the Jacobian of the transformation from the variable k_i to $\tilde{k}_i$ $i \in 1,\ldots,n$ is equal to 1 (the transformation is defined by a triangular matrix). After substitution of the expressions (28) into formula (27) and a number of transformations, we obtain the following

$$\begin{aligned} P(r,r_0,t-t_0) \approx e^{-\Delta_{n+1}r}\int_{-\infty}^{\infty}\prod_{i=i}^{n}\frac{1}{\tilde{k}_i^2+1}\exp\Big(-\frac{\sigma^2}{2}k_{n+1}^2(t-t_0)\Big)\exp\Big(-\frac{\sigma^2}{2}\big(2Ak_{n+1}+B\big)\Big) \\ \times \sin(k_{n+1}r)\,\sin\big((k_{n+1}+\rho_1)r_0\big)\,\frac{dk_{n+1}}{\pi}\prod_{i=i}^{n}\frac{d\tilde{k}_i}{\pi}. \end{aligned} \quad (29)$$

The following denotations where introduced

$$A = \sum_{i=1}^{n}\Delta_i\rho_i, \quad B = \sum_{i=1}^{n}\Delta_i\rho_i^2, \quad \rho_i = \sum_{j=i}^{n}\tilde{k}_j\Delta_j. \quad (30)$$

Integrations over k_{n+1} in (29) we perform in closed form. Next, we proceed to the limit $\Delta_i \to 0$, $i \in \{1,\ldots,n\}$, $n \to \infty$. Hereby, the points k_i, t_i, $i \in 1,\ldots,n$ are considered to be values of some curve $k(\tau)$, $\tau \in [t_0,t]$. Sums in (30) acquire a sense of integral sums and in the limit they define integrals for functions of a curve $k(\tau), \tau \in [t_0,t]$. To sum up, we obtain the following representation via path integral

$$P(r,r_0,t-t_0) = \int_{-\infty}^{\infty}\mathcal{D}k(\tau)\exp\Big(-\frac{\sigma^2}{2}B(k(\tau)) + i\rho\big(k(\tau)\big)r_0\Big)\,K_D\big(r,r_0+i\sigma^2A(k(\tau)),t-t_0\big). \quad (31)$$

The following quantities where denoted:

$$\begin{aligned} A(k) &= \int_{t_0}^{t}\Big(\int_{\tau}^{t}k(\tau')\,d\tau'\Big)d\tau, \quad B(k) = \int_{t_0}^{t}\Big(\int_{\tau}^{t}dk(\tau')\,\tau'\Big)^2 d\tau, \\ \rho(k) &= \int_{t_0}^{t}k(\tau)\,d\tau, \quad \mathcal{D}k(\tau) = \prod_{\tau=t_0}^{t}\frac{dk(\tau)}{\pi(k(\tau)^2+1)}. \end{aligned} \quad (32)$$

Integration is performed with the functional measure normalized to 1

$$\int_{-\infty}^{\infty}\mathcal{D}k(\tau) = 1. \quad (33)$$

Let us transform the path integral (31), in order to give it a traditional form. It can be seen from (31) and (32) that the exponents of integral expression contain linear and quadratic functionals of $k(t)$. As the next step, let us separate quadratic terms in order to linearize them. It is easy to see that in exponents of integral expression (31) one can separate the following terms:

$$\begin{gathered} -\frac{\sigma^2}{2}\left(B(k)-\frac{A(k)^2}{t-t_0}\right)=-\frac{\sigma^2}{2}\int_{t_0}^{t}Q(\tau)^2d\tau, \\ Q(\tau)=\int_{\tau}^{t}k(\tau')\,d\tau'-\frac{1}{t-t_0}\int_{t_0}^{t}\left(\int_{\tau}^{t}k(\tau')\,d\tau'\right)d\tau. \end{gathered} \tag{34}$$

Then the term in exponent (34) we can linearize with the help of Gaussian quadrature [12]

$$\begin{gathered} \exp\left(-\frac{\sigma^2}{2}\int_{t_0}^{t}Q(\tau)^2d\tau\right)=\int\mathcal{D}q(\tau)\exp\left(-\frac{1}{2}\int_{t_0}^{t}q(\tau)^2d\tau\right)\exp\left(i\sigma\int_{t_0}^{t}Q(\tau)\,q(\tau)\,d\tau\right), \\ \mathcal{D}q(\tau)=\prod_{\tau=t_0}^{t}\frac{dq(\tau)}{\sqrt{2\pi dt}}. \end{gathered} \tag{35}$$

Using Gaussian quadrature (35) and path integral

$$\int_{-\infty}^{\infty}\mathcal{D}k(\tau)\exp\left(i\int_{t_0}^{t}k(\tau)\,\phi(\tau)\,d\tau\right)=\exp\left(-\int_{t_0}^{t}|\phi(\tau)|\,d\tau\right) \tag{36}$$

after transformations we receive a formula for discounting multiplier

$$P(r_0,t-t_0)=\int_{0}^{\infty}P(r,r_0,t-t_0)\,dr, \tag{37}$$

$$P(r,r_0,t-t_0)=P_0(r,r_0,t-t_0)-P_0(-r,r_0,t-t_0), \tag{38}$$

with the following notations

$$\begin{aligned} P_0(r,r_0,t-t_0)=\int_{-\infty}^{\infty}&\mathcal{D}q(\tau)\exp\left(-\frac{1}{2}\int_{t_0}^{t}q(\tau)^2d\tau\right)\exp\left(-\int_{t_0}^{t}\left|r-\sigma\int_{\tau}^{t}q(\tau')\,d\tau'\right|d\tau\right) \\ &\times\delta\left(r-r_0-\sigma\int_{t_0}^{t}q(\tau)d\tau\right). \end{aligned} \tag{39}$$

The path integral is written according to quantum-mechanic terminology, in the space of velocities [7,8], where $r(\tau)$ is coordinate, and $\sigma q(\tau)$ is velocity. Based on (39), let us transit to an equivalent coordinate representation

$$\begin{gathered} P_0(r,r_0,t-t_0)=\int_{r_0}^{r}\mathcal{D}r(\tau)\exp\left(-\frac{1}{2\,\sigma^2}\int_{t_0}^{t}\dot{r}(\tau)^2d\tau\right)\exp\left(-\int_{t_0}^{t}|r(\tau)|\,d\tau\right), \\ \mathcal{D}r(\tau)=\prod_{\tau=t_0}^{t}\frac{dr(\tau)}{\sqrt{2\pi\sigma^2dt}}. \end{gathered} \tag{40}$$

Therefore, it has been found out that the expressions (38), (39) or (40) determine a discounting multiplier and term structure of interest rate in Merton model ($\mu=0$) with Dirichlet condition at the boundary $r=0$.

Let us note that the path integral (40) has the same structure as (13) (for $\mu=0$) only it contains a multiplier in the second exponential term. The integral measure is defined for the entire range $-\infty<r(\tau)<\infty$, $\tau\in[t_0,t]$. Integration on half-plane occurs in the formula (37).

There are approaches developed for problems of quantum mechanics which take into account boundary conditions in path integral method. In particular, for propagator of Schrödinger equations given by path integral, the boundary conditions are given with the help of point interaction using δ-functions and its derivatives [15, 16]. Then for Green's function (a Fourier transform of propagator over time variable) with boundary condition we receive an algebraic equation which expresses it through Green's function without boundary condition. Many results received for Green's functions can be carried over to the case of stochastic problems, if Laplace transform is consider instead of Fourier transform. If to consider the path integral (15) then according to (51) the following Laplace transform

$$P(r,r_0,s)=\int_0^{\infty}P(r,r_0,t)\,e^{-s\,t}\,dt \tag{41}$$

does not exist for the case of the Merton model. For the Vasicek model (19) Laplace transform according to (57) does not exist for entire parameters range. That is why, the considered methods of accounting for boundary conditions are not applicable in this case.

4. Asymptote of term structure of interest rate in Merton model

Path integral in the form (40) has been explored in problems of quantum mechanics where it defines the propagator of particle motion. In particular, in [16] an expressions for Fourier transform are given for the mentioned propagator (Green's function). In our case, based on expression for Green's function we can compare Laplace transform for $P_0(r,r_0,t)$ (40) (in this case Laplace transform exists). However, obtained by means of this approach a formally precise expression for $P_0(r,r_0,s)$ is impractical. Based on it, it is difficult to inverse it to the original domain.

Let us consider the asymptotic estimate of the model under discussion. In particular, based on formulas (15), (38) and (40) let us write the following

$$P(r_0,t-t_0)=\int_0^{\infty}P(r,r_0,t-t_0)\,dr<\int_0^{\infty}P_0(r,r_0,t-t_0)\,dr+\int_0^{\infty}P_0(-r,r_0,t-t_0)\,dr. \tag{42}$$

Accordingly, for each term in (42), one can obtain an upper bound estimate while taking into account inequality in the path integral (40)

$$\exp\left(-\int_{t_0}^{t}|r(\tau)|\,d\tau\right)\leqslant\exp\left(-\left|\int_{t_0}^{t}r(\tau)\,d\tau\right|\right). \tag{43}$$

An exponent in the right part of the inequality (43) can be represented with the help of the following integral transform

$$\exp\left(-\left|\int_{t_0}^{t}r(\tau)\,d\tau\right|\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}\exp\left(-i\omega\int_{t_0}^{t}r(\tau)\,d\tau\right)d\omega. \tag{44}$$

To sum up, using the formulae (42), (43) and (44) and after some transforms, we obtain the following expression for estimation

$$P(r_0,t-t_0)\leqslant\frac{1}{2}e^{-r_0(t-t_0)+\frac{1}{6}\sigma^2(t-t_0)^3}\left[1+\operatorname{erf}\left(\frac{3r_0-\sigma^2(t-t_0)^2}{\sqrt{6}\sigma\sqrt{t-t_0}}\right)+e^{2r_0(t-t_0)}\operatorname{erfc}\left(\frac{3r_0+\sigma^2(t-t_0)^2}{\sqrt{6}\sigma\sqrt{t-t_0}}\right)\right]. \tag{45}$$

For $\sigma\sqrt{t-t_0}\gg 1$ based on (45), the main term of the asymptotic can be determined

$$P(r_0,t-t_0)\leqslant\frac{\sqrt{6}}{\sqrt{\pi\sigma^2(t-t_0)^3}}\exp\left(-\frac{3r_0^2}{2\sigma^2(t-t_0)}\right). \tag{46}$$

That means that the average value in Merton models with Dirichlet boundary condition is limited. In the same time, for an ordinary Merton model, the average value is unlimited (51).

5. Conclusions

In the current work, the application of path integral method to some stochastic models of interest rate has been considered. Boundary conditions have been also introduced in order to limit the variables domain. The well-known stochastic models of Merton and Vasicek that are used to describe term structure of interest rates have been examined. These models describe incorrectly the term structure of interest rate which leads to negative values of interest rate. The known results for term structure of interest rate in Merton and Vasicek models are obtained using path integral method by two approaches. According to the first approach, the path integral uses Wieners measure with the following substitution of solutions of stochastic equations of the models. According to the second approach, a variable substitution is performed in path integral with Wieners measure and transformation to integral measure related to the stochastic variable by Merton and Vasicek equations. It is easier to introduce boundary conditions into the path integral method within the framework of the second approach. For that matter, during calculation of average values one needs to set the transition probability which satisfies boundary conditions. Calculations are carried out on example of Merton model with a zero drift $\mu = 0$. For the discounting multiplier which determines the term structure of interest rate, the path integral representation has been obtained. Although for obtained path integrals a formally precise Laplace transform over the time variable can be established, it is fairly complicated and impractical. The work presents an estimate of discounting multiplier and shows that it is limited. A more detailed research of obtained path integrals is going to be carried out in a separate work, where other cases of boundary conditions in Merton and Vasicek models will be examined.

A. Appendix

A.1. Merton model

For a symmetry reason let us write down the expression (6) in the following form

$$P(t-t_0) = \int_{-\infty}^{\infty} P(w, t-t_0)\, dw, \tag{47}$$

$$P(w, t-t_0) = \int_{w_0}^{w} \mathcal{D}w(\tau)\, \exp\Big(-\int_{t_0}^{t} r(\tau)\, d\tau\Big). \tag{48}$$

Instead of $r(\tau)$ in (48) let us substitute solution (10). For $P(w, t-t_0)$, after transformations we receive the following

$$\begin{aligned} P(w, t-t_0) &= P_0(t-t_0) \int_{w_0}^{w} \mathcal{D}w(\tau) \exp\Big(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w(\tau)}{d\tau}\Big)^2 d\tau\Big) \exp\Big(-\sigma\int_{t_0}^{t}\big(w(\tau)-w_0\big)\, d\tau\Big), \\ P_0(t-t_0) &= \exp\Big(-r_0(t-t_0) - \frac{1}{2}\mu(t-t_0)^2\Big). \end{aligned} \tag{49}$$

The path integral in (49) equals to [7,8]

$$\begin{aligned} &\int_{w_0}^{w} \mathcal{D}w(\tau) \exp\Big(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w(\tau)}{d\tau}\Big)^2 d\tau\Big) \exp\Big(-\sigma\int_{t_0}^{t}(w(\tau)-w_0) d\tau\Big) \\ &\qquad = \int_{0}^{w-w_0} \mathcal{D}w'(\tau) \exp\Big(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w'(\tau)}{d\tau}\Big)^2 d\tau\Big) \exp\Big(-\sigma\int_{t_0}^{t}(w'(\tau))\, d\tau\Big) \\ &= \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} \exp\Big(-\frac{1}{2\sigma^2}\frac{(w-w_0)^2}{t-t_0}\Big) \exp\Big(-\frac{1}{2}(w-w_0)(t-t_0)\Big) \exp\Big(\frac{\sigma^2}{24}(t-t_0)^3\Big). \end{aligned} \tag{50}$$

After integrating over w we receive

$$P(t-t_0)=\exp\Big(-r_0(t-t_0)-\frac{1}{2}\mu(t-t_0)^2+\frac{\sigma^2}{6}(t-t_0)^3\Big), \tag{51}$$

$$r(t)=r_0+\frac{1}{2}\mu(t-t_0)-\frac{\sigma^2}{6}(t-t_0)^2. \tag{52}$$

As can be seen from (52) in case when stochastic term is major, the interest rate r has negative values.

A.2. Vasicek model

Let us rewrite the expression (6) in the same way as in the cases with (47) and (48). Instead of $r(\tau)$, let us substitute the solution (11) into (48). For $P(w,t-t_0)$, after a number of transformations we obtain the following:

$$P(w,t-t_0)=P_0(t-t_0)\int_{w_0}^{w}\mathcal{D}w(\tau)\exp\Big(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w(\tau)}{d\tau}\Big)^2d\tau\Big)\exp\Big(-\sigma\int_{t_0}^{t}B(t-\tau)\frac{dw(\tau)}{d\tau}\,d\tau\Big), \tag{53}$$

$$P_0(t-t_0)=\exp\big(-B(t-t_0)(r_0-\mu)-\mu(t-t_0)\big),\quad B(\tau)=\frac{1}{\beta}\big(1-e^{-\beta\tau}\big). \tag{54}$$

The path integral (53) is easy to calculate, so as a result we receive the following

$$\begin{gathered}\int_{w_0}^{w}\mathcal{D}w(\tau)\exp\Big(-\frac{1}{2}\int_{t_0}^{t}\Big(\frac{w(\tau)}{d\tau}\Big)^2d\tau\Big)\exp\Big(-\int_{t_0}^{t}B(t-\tau)\frac{dw(\tau)}{d\tau}d\tau\Big)\\=\frac{1}{\sqrt{2\pi(t-t_0)}}\exp\Big(-\frac{1}{2(t-t_0)}\Big(w-w_0+\sigma\int_{t_0}^{t}B(t-\tau)d\tau\Big)^2\Big)\exp\Big(\frac{\sigma^2}{2}\int_{t_0}^{t}B(t-\tau)^2d\tau\Big).\end{gathered} \tag{55}$$

After substituting (55) into expression (53) and calculating integral over w in (47) we receive the following for average values in Vasicek model

$$P(t-t_0)=P_0(t-t_0)\exp\Big(\frac{\sigma^2}{2}\int_{t_0}^{t}B(t-\tau)^2d\tau\Big). \tag{56}$$

The expression (56) coincides with the well-known results for term structure of interest rate for the Vasicek model [1] (one needs to apply the following notation $t\to T$, $t_0\to\tau$). As it is accepted in models of term structure of interest rate that time T is fixed and (56) is considered for various initial values of τ.

As it was already noted, the value (56) in Vasicek model shows an asymptotic behavior that does not agree with the sense of interest rate. It is not hard to show that for $\sigma^2(t-t_0)\gg 1$, $\beta(t-t_0)\gg 1$ we can receive

$$P(t-t_0)\approx\exp\Big(-\frac{1}{2}(t-t_0)\Big(2\mu-\frac{\sigma^2}{\beta^2}\Big)\Big) \tag{57}$$

or

$$r(t)\approx\mu-\frac{\sigma^2}{2\beta^2}. \tag{58}$$

As can be seen for condition $\frac{\sigma^2}{2\beta^2}>\mu$ the interest rate becomes negative. It means that in case of dominating of stochastic term in the Vasicek equation, the model incorrectly describes the dynamics of interest rate. One can see as well that the term with β plays a kind of stabilizing role in stochastic dynamics when compared to Merton model.

B. Appendix

For Merton model the transition probability is equal to

$$K(r, r_0, t - t_0) = \int_{r_0}^{r} \mathcal{D}r \exp\left(-\frac{1}{2\sigma^2} \int_{t_0}^{t} \left(\frac{dr}{d\tau} - \mu \right)^2 d\tau \right), \tag{59}$$

which after integrations leads to a known result [7, 8]

$$K(r, r_0, t - t_0) = \frac{1}{\sqrt{2\pi\sigma^2(t - t_0)}} \exp\left(-\frac{1}{2} \frac{(r - r_0 - \mu(t - t_0))^2}{\sigma^2(t - t_0)} \right). \tag{60}$$

In Vasicek model, after integrating

$$K(r, r_0, t - t_0) = \exp\left(\frac{1}{2}\beta(t - t_0) \right) \int_{r_0}^{r} \mathcal{D}r \exp\left(-\frac{1}{2\sigma^2} \int_{t_0}^{t} \left(\frac{dr}{d\tau} - \beta\big(\mu - r(\tau)\big) \right)^2 d\tau \right), \tag{61}$$

the following is received for the transition probability

$$K(r, r_0, t - t_0) = \exp\left(\frac{1}{2}\beta(t - t_0) \right) \exp\left(-\frac{\beta^2}{2\sigma^2}\big((r - \mu)^2 - (r_0 - \mu)^2\big) \right) \tag{62}$$
$$\times \sqrt{\frac{\beta}{2\pi\sigma^2 \sinh(\beta(t - t_0))}} \exp\left(-\frac{\beta}{2\sigma^2} \frac{\big((r - \mu)^2 + (r_0 - \mu)^2\big)\cosh(\beta(t - t_0)) - 2(r - \mu)(r_0 - \mu)}{\sinh(\beta(t - t_0))} \right).$$

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Метод функціонального інтегрування в моделях відсоткових ставок

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Розглянуто метод функціонального інтегрування в стохастичних моделях Мертона та Васічека відсоткової ставки. Продемонстровано побудову функціональних інтегралів двома способами: перший — за мірою Вінера з підстановкою розв'язків стохастичних рівнянь для моделей; другий — перехід від міри Вінера до міри інтегрування пов'язаної зі стохастичними змінними рівнянь Мертона та Васічека. Розглянуто введення граничних умов у другому способі для усунення некоректних часових асимптотик класичних моделей Мертона та Васічека відсоткових ставок. На прикладі моделі Мертона з нульовим дрейфом розглянуто граничну умову Діріхлє. Отримано представлення функціональним інтегралом для часової структури відсоткової ставки. Наведено оцінку отриманих функціональних інтегралів, де показано, що часова асимптотика є обмеженою.

Ключові слова: *відсоткова ставка, стохастична модель, умовна ймовірність, функціональний інтеграл.*