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Asymptotic analysis of the Korteweg–de Vries equation by the nonlinear WKB technique

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The paper deals with the Korteweg–de Vries equation with variable coefficients and a small parameter at the highest derivative. The non-linear WKB technique has been used to construct the asymptotic step-like solution to the equation. Such a solution contains regular and singular parts of the asymptotics. The regular part of the solution describes the background of the wave process, while its singular part reflects specific features associated with soliton properties. The singular part of the searched asymptotic solution has the main term that, like the soliton solution, is the quickly decreasing function of the phase variable τ. In contrast, other terms do not possess this property. An algorithm of constructing asymptotic step-like solutions to the singularly perturbed Korteweg–de Vries equation with variable coefficients is presented. In some sense, the constructed asymptotic solution is similar to the soliton solution to the Korteweg–de Vries equation $u_t + uu_x + u_{xxx} = 0$. Statement on the accuracy of the main term of the asymptotic solution is proven.

Keywords: *Korteweg–de Vries equation, nonlinear WKB technique, asymptotic analysis, singular perturbation, soliton, asymptotic solution.*

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1. Introduction

In 1895 the Korteweg–de Vries equation (the KdV equation) of the form

$$u_t + \alpha u u_x + u_{xxx} = 0, \quad \alpha \in \mathbf{R}\setminus\{0\}, \tag{1}$$

was proposed for mathematical description of solitary waves [1,2]. This is currently a well-known non-linear equation of modern theoretical and mathematical physics since it possesses solutions of different kinds including soliton ones [3].

In 1965 this equation attracted the attention of many scholars after a mathematical explanation [4] of the elastic interaction of the so-called soliton waves described by equation (1). From this moment to the present, the KdV equation has been the subject of deep research in mathematics and applications in the mathematical modeling of different complex systems in various fields of science and technology, including hydromechanics, solid state theory, plasma, biology, economic systems, telecommunications, optical and information systems, and others [5,6].

Over the past 50 years, many monographs and thousands of articles have been published on a wide range of problems for the KdV equation and its generalizations. First of all, the attention of researchers was focused on the construction of exact solutions of the KdV equation, in particular, soliton and finite-gap solutions. The Cauchy problem with initial data from various functional spaces, including the space of quickly decreasing functions $\mathcal{S}(\mathbf{R})$ was also considered [7–10]. In addition, scientists have been analyzed the issues of integrability of the KdV equation. These studies used both methods of functional analysis, in particular, of spectral theory, and methods of Hamiltonian mechanics, symplectic and symmetry analysis, Backlund and Darboux transformations, and many others [11–15].

It should also be noted that due to algebraic-geometric methods that are effectively used in the study of the KdV equation, new integrable nonlinear dynamical systems have been found [16].

For the KdV equation, a number of problems in the presence of small perturbations are also considered. In particular, H. Flaschka, M. G. Forest and D. McLaughlin [17] applied the averaging technique for studying the problem of modulation of nonlinear waves described by the KdV equation. Later a similar problem was studied for the nonlinear Schrödinger equation [12, 18]. This problem is called the problem of ergodic deformations for completely integrable dynamical systems because although the equation under consideration does not contain a small parameter ε explicitly, but the dependence on the parameter ε of finite-gap solutions of the equation is displayed through small deformations of the associated Riemann manifolds.

Asymptotic methods proved to be an effective tool for solving other problems of the KdV equation. The study of the singularly perturbed KdV equation by asymptotic methods was started by R. M. Miura and M. D. Kruskal [19], who in 1974 generalized the linear WKB technique to the case of nonlinear equations and constructed the main term of the asymptotic expansion for the quasiperiodic solution to the KdV equation of the following form

$$u_t + uu_x + \delta^2 u_{xxx} = 0,$$

where δ is a small parameter. This approach is called the nonlinear WKB technique [20, 21].

Later, P. D. Lax and S. D. Levermore studied the problem on the limit of the solution to the Cauchy problem for the singularly perturbed KdV equation as a small parameter tends to zero [22–24]. Under solving the problem they used the method of the inverse scattering transform approach while the eigenfunctions of the corresponding Schrödinger operator with singular perturbation were approximately constructed by the linear WKB technique. Currently, existing methods for studying nonlinear partial differential equations with variable coefficients do not allow us to find their solutions explicitly as a rule. Therefore, perhaps the only effective approach for constructing approximate solutions of such equations with a small perturbation is asymptotic methods.

One of the first papers devoted to singularly perturbed nonlinear partial differential equations with variable coefficients is the paper by V. P. Maslov and his students [20], where, using the nonlinear WKB technique, the authors constructed an asymptotic solution, the structure of which is in some sense similar to the structure of a soliton solution. Such solutions are called *asymptotic soliton-like solutions* [21], because in the particular case of constant coefficients, the obtained solutions coincide with the soliton solutions.

The above model describes the motion of a fluid in a medium with variable depth and shows that to determine the wave processes in media with variable characteristics and small dispersion, it is necessary to study singularly perturbed partial differential equations with variable coefficients. For the singularly perturbed KdV equation with variable coefficients and its generalizations, general algorithms for constructing asymptotic soliton-like solutions are proposed, and their justification is given in [25–27].

The paper deals with the KdV equation with variable coefficients and a small parameter at the highest derivative of the following form

$$\varepsilon\, u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u\, u_x. \tag{2}$$

Here

$$a(x,t,\varepsilon) = \sum_{k=0}^{N} a_k(x,t)\varepsilon^k + \mathrm{O}\big(\varepsilon^{N+1}\big), \quad b(x,t,\varepsilon) = \sum_{k=0}^{N} b_k(x,t)\varepsilon^k + \mathrm{O}\big(\varepsilon^{N+1}\big),$$

where $a_k(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, $b_k(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, $k=\overline{0,N}$, $T>0$, ε is a small parameter, and the notation $\Psi(x,t,\varepsilon) = \mathrm{O}\big(\varepsilon^N\big)$ as $\varepsilon\to 0$ is used. It means that for any compact set $K\subset \mathbf{R}\times[0;T]$ there exist $\varepsilon_0>0$ and a constant $C>0$ such that the inequality $|\Psi(x,t,\varepsilon)| \leqslant C\,\varepsilon^N$ takes place for all $(x,t)\in K$, $\varepsilon\in(0;\varepsilon_0)$, where constant C depends only on N and K.

Equation (2) arises during the mathematical simulating wave processes in inhomogeneous media with a small dispersion [28]. Its analysis is a rather non-trivial mathematical problem due to the presence of nonlinearity and variable coefficients. We study the equation by asymptotic techniques [29,30] that provide us with constructing its approximate solutions [31, 32] as well as their analysis. The solutions contain regular and singular parts. It should be also mentioned that the regular part of the solution describes the background of the wave process while its singular part reflects specific characteristics associated with soliton properties.

These asymptotic solutions have also certain features. Taking into account them it's possible to identify two types of the solutions. The solutions of the first type have singular part all terms of which belong to the space of quickly decreasing functions with respect to the so-called phase variable τ. These solutions are called *the asymptotic solutions of soliton type* [30]. Solutions of the other type have a singular part the main term of which is the quickly decreasing function of the phase variable while the other terms have no this property. It means that higher terms of the singular part of the asymptotic tend to zero as $\tau \to +\infty$ and tend to non-zero as $\tau \to -\infty$. Such asymptotic solutions are called *the asymptotic step-like solutions* [33].

In the paper we consider the problem of constructing asymptotic step-like solutions to singularly perturbed equation (2) and propose an algorithm of obtaining them. We also present the general form of the regular terms of the asymptotic step-like solutions to the singularly perturbed KdV equation (2).

2. Preliminary notes and definitions

The asymptotic soliton-like solutions to equation (2) are found by the non-linear WKB technique through the algorithm described in [25,29]. Accordingly properties of step-like solutions the asymptotic expansion has a special behavior at large arguments. Therefore the terms of the asymptotic solution have to belong to some functional spaces.

Let $G_1 = G_1(\mathbf{R} \times [0;T] \times \mathbf{R})$ be a linear space of infinitely differentiable functions $f(x,t,\tau)$, $(x,t,\tau) \in \mathbf{R} \times [0;T] \times \mathbf{R}$, such that uniformly with respect to variables (x,t) on any compact $K \subset \mathbf{R} \times [0;T]$ for all non-negative integer numbers n, m, p, q the following conditions hold [21]:

1) the relation

$$\lim_{\tau \to +\infty} \tau^n \frac{\partial^{m+p+q}}{\partial x^m \partial t^p \partial \tau^q} f(x,t,\tau) = 0, \quad (x,t) \in K,$$

is fulfilled;

2) there exists an infinitely differentiable function $f^-(x,t)$ such that

$$\lim_{\tau \to -\infty} \tau^n \frac{\partial^{m+p+q}}{\partial x^m \partial t^p \partial \tau^q} (f(x,t,\tau) - f^-(x,t)) = 0, \quad (x,t) \in K,$$

is satisfied.

Let $G_0 = G_0(\mathbf{R} \times [0;T] \times \mathbf{R}) \subset G_1$ be a space of functions $f(x,t,\tau)$ such that uniformly with respect to variables (x,t) on any compact $K \subset \mathbf{R} \times [0;T]$ the condition

$$\lim_{\tau \to -\infty} f(x,t,\tau) = 0, \quad (x,t) \in K,$$

holds [21].

The asymptotic step-like solution to equation (2) is searched in the form

$$u(x,t,\varepsilon) = Y_N(x,t,\tau,\varepsilon) + \mathrm{O}\big(\varepsilon^{N+1/2}\big), \tag{3}$$

where

$$Y_N(x,t,\tau,\varepsilon) = U_N(x,t,\varepsilon) + V_N(x,t,\tau,\varepsilon), \quad \tau = \frac{x - \varphi(t)}{\sqrt{\varepsilon}},$$

$$U_N(x,t,\varepsilon)=\sum_{j=0}^{2N}\varepsilon^{j/2}u_j(x,t),\quad V_N(x,t,\tau,\varepsilon)=\sum_{j=0}^{2N}\varepsilon^{j/2}V_j(x,t,\tau).$$

Here coefficients $u_j(x,t)\in C^\infty(\mathbf{R}\times[0;T])$, $V_0(x,t,\tau)\in G_0$, $V_j(x,t,\tau)\in G_1$, $j=\overline{1,2N}$.

The function $x=\varphi(t)$ is called phase function of the asymptotic step-like solution and τ is a phase variable. The function $U_N(x,t,\varepsilon)$ is called regular part and the function $V_N(x,t,\tau,\varepsilon)$ is called singular part of asymptotic solution (3). We need to determine both the regular and the singular part of the asymptotic as well as the phase function.

3. Constructing the asymptotic solution

To construct the asymptotic step-like solution to equation (2) we firstly deduce partial differential equations for the terms of the regular and the singular parts, and then consider the problem of constructing their solutions in some functional spaces that are described above. We solve differential equations for the regular terms, since they are independent on the singular terms. Later we study the problem of constructing the singular terms. The task of determining the singular part of the asymptotic is rather complicated and consists of several stages, which are described in details below.

3.1. The regular part of the asymptotics

Differential equations for the terms of asymptotic expansion (3) we find in the following way. Remind that the terms of the singular part belong to the space G_1. Taking into account the property we substitute asymptotic expansion (3) into equation (2) and calculate limits as $\tau\to+\infty$. In such a way we find differential equation for the regular part $U_N(x,t,\varepsilon)$. Splitting the equation we obtain system of relations for the terms of the regular part of asymptotic (3) of the following form:

$$a_0(x,t)\frac{\partial u_0}{\partial t}+b_0(x,t)u_0\frac{\partial u_0}{\partial x}=0, \tag{4}$$

$$a_0(x,t)\frac{\partial u_j}{\partial t}+b_0(x,t)u_0\frac{\partial u_j}{\partial x}+b_0(x,t)u_j\frac{\partial u_0}{\partial x}=F_j(x,t),\quad j=\overline{1,2N}, \tag{5}$$

where functions $F_j(x,t)\in C^\infty(\mathbf{R}\times[0;T])$, $j=\overline{1,2N}$, are recursively defined through functions $u_0(x,t)$, $u_1(x,t)$, …, $u_{j-1}(x,t)$, $j=\overline{1,2N}$.

Firstly remark that system (4), (5) has trivial solution $u_j(x,t)=0$, $j=\overline{1,2N}$. In the case asymptotic soliton-like solution has zero background [21].

The general solution of system (4), (5) can be found in implicit form by means of method of characteristics because (4) is the Hopf equation with variable coefficients and (5) are the first order non-homogeneous linear equations with variable coefficients.

Lemma 1. *Let $a_0(x,t)$, $b_0(x,t)$ be continuous and satisfy the inequality $a_0(x,t)b_0(x,t)\neq 0$ for all $(x,t)\in\mathbf{R}\times[0;T]$, a function $\psi_0(x,t,c_1)=c_2$ is the first integral of differential equation*

$$a_0(x,t)\,dx-c_1\,b_0(x,t)\,dt=0,\ c_1\in\mathbf{R}.$$

Then the general solution to equation (4) can be written as

$$\Phi_0(u_0,\psi_0(x,t,u_0))=0, \tag{6}$$

where a function $\Phi_0(\xi,\eta)\in C^{(1)}(\Xi_0;\mathbf{R})$ for a domain $\Xi_0\subset\mathbf{R}^2$ containing at least one point $(\xi_0,\eta_0)\in\Xi_0$ such that $\Phi_0(\xi_0,\eta_0)=0$ and

$$\frac{d\Phi_0(u_0,\psi_0(x,t,u_0))}{du_0}\neq 0\ \ \text{for all}\ \ (x,t,u_0)\in\Omega_0.$$

Here domain Ω_0 is a set of variables (x,t,u_0) for which the mapping $\Omega_0 \ni (x,t,u_0) \to (u_0, \psi_0(x,t,u_0)) \in \Xi_0$ is defined.

Proof. It is well known that the general solution of the first order quasi-linear partial differential equation is represented through complete system of its independent first integrals. For the equation (4) the first integrals are determined from the equations of characteristics of the following form

$$\frac{dt}{a_0(x,t)} = \frac{dx}{u_0 b_0(x,t)} = \frac{du_0}{0}. \tag{7}$$

System (7) has the first integral in the form $u_0(x,t) = c_1$, where c_1 is an arbitrary real. The other first integral is defined from the first order non-linear ordinary differential equation

$$\frac{dx}{dt} = \frac{a_0(x,t)}{u_0 b_0(x,t)}, \tag{8}$$

where we have to put $u_0 = c_1$. For any $c_1 \in \mathbf{R}$ it has a local solution $x = x(t, c_1)$, $t \in (t_0 - \delta; t_0 + \delta)$, under initial condition $x_0 = x(t_0, c_1)$. So, the other first integral for system (7) can be represented as $\psi_0(x,t,c_1)$, where $\psi_0(x,t,c_1)$ is a solution to equation (8) under condition $u_0 = c_1$.

Thus, the general solution to (4) is given by formula (6).

The lemma is proven. ■

Lemma 2. *Let the functions $a_0(x,t)$, $b_0(x,t)$ be continuous, function $u_0(x,t)$ is a solution to equation (4) and they satisfy the inequality $a_0(x,t) b_0(x,t) u_0(x,t) \neq 0$ for all $(x,t) \in \mathbf{R} \times [0;T]$, a function $h(x,t)$ is the first integral of differential equation*

$$a_0(x,t)\, dx - b_0(x,t)\, u_0(x,t)\, dt = 0.$$

Then for any $j \in \mathbf{N}$ the general solution to (5) can be written as

$$\Phi_j(h(x,t), \psi_j(x,t,u_j)) = 0, \tag{9}$$

where a function $\Phi_j(\xi,\eta) \in C^{(1)}(\Xi_j; \mathbf{R})$ for a domain $\Xi_j \subset \mathbf{R}^2$ containing at least one point $(\xi_{0j}, \eta_{0j}) \in \Xi_j$ such that $\Phi_j(\xi_{0j}, \eta_{0j}) = 0$ and

$$\frac{d\Phi_j(h(x,t), \psi_j(x,t,u_j))}{du_j} \neq 0 \quad \text{for all} \quad (x,t,u_j) \in \Omega_j.$$

Here the domain Ω_j is a set of variables (x,t,u_j) for which the mapping

$$\Omega_j \ni (x,t,u_j) \to (h(x,t), \psi_j(x,t,u_j)) \in \Xi_j$$

is defined.

In representation (9) the function $\psi_j = \psi_j(x,t,u_j)$ is given by formula

$$\psi_j(x,t,u_j) = [u_j - B_j(x,t)]\, e^{A(x,t)},$$

where

$$A(x,t) = \int_{x_0}^{x} \frac{u_{0x}(\vartheta, g(\vartheta, h(x,t)))}{u_0(\vartheta, g(\vartheta, h(x,t)))}\, d\vartheta,$$

$$B_j(x,t) = \int_{x_0}^{x} e^{A(\vartheta,t) - A(x,t)} \frac{F_j(\vartheta, g(\vartheta, h(x,t)))}{b_0(\vartheta, g(\vartheta, h(x,t))) u_0(\vartheta, g(\vartheta, h(x,t)))}\, d\vartheta,$$

and the function $g(x,\alpha)$ is implicitly defined by the relation $h(x, g(x,\alpha)) = \alpha$.

Proof. Similar to proving Lemma 1 we consider the equations of characteristics for (5) in the following form

$$\frac{dt}{a_0(x,t)} = \frac{dx}{u_0 b_0(x,t)} = \frac{du_j}{F_j(x,t) - b_0(x,t)u_{0x}u_j}.$$

It is clear that nonlinear ordinary differential equation

$$\frac{dx}{dt} = \frac{u_0(x,t)b_0(x,t)}{a_0(x,t)},$$

has the first integral denoted by $h(x,t)$.

The other equation of characteristics

$$\frac{du_j}{dx} = -\frac{u_{0x}}{u_0}\,u_j + \frac{F_j(x,t)}{u_0 b_0(x,t)}$$

is the first order linear ordinary differential equation whose solution we can represent by means of quadrature as follows

$$\psi_j(x,t,u_j) = [u_j - B_j(x,t)]\, e^{A(x,t)},$$

where

$$A(x,t) = \int_{x_0}^{x} \frac{u_{0x}(\vartheta, g(\vartheta, h(x,t)))}{u_0(\vartheta, g(\vartheta, h(x,t)))}\, d\vartheta,$$

$$B_j(x,t) = \int_{x_0}^{x} e^{A(\vartheta,t)-A(x,t)} \frac{F_j(\vartheta, g(\vartheta, h(x,t)))}{b_0(\vartheta, g(\vartheta, h(x,t)))u_0(\vartheta, g(\vartheta, h(x,t)))}\, d\vartheta,$$

and the function $g(x,\alpha)$ is defined by the relation $h(x, g(x,\alpha)) = \alpha$ in implicit way.

So, for any $j \in \mathbf{N}$ the general solution to equation (5) is written in implicit form as

$$\Phi_j\left(h(x,t), \psi_j(x,t,u_j)\right) = 0,$$

where a function $\Phi_j(\xi,\eta) \in C^{(1)}(\Xi_j; \mathbf{R})$ for a domain $\Xi_j \subset \mathbf{R}^2$ containing at least one point $(\xi_{0j}, \eta_{0j}) \in \Xi_j$ such that $\Phi_j(\xi_{0j}, \eta_{0j}) = 0$ and

$$\frac{d\Phi_j(h(x,t), \psi_j(x,t,u_j))}{du_j} \neq 0 \quad \text{for all} \quad (x,t,u_j) \in \Omega_j.$$

Here the domain Ω_j is a set of variables (x,t,u_j) for which the mapping

$$\Omega_j \ni (x,t,u_j) \to (h(x,t), \psi_j(x,t,u_j)) \in \Xi_j$$

is defined.

The lemma is proven. ■

Now we proceed to determining the singular part of the asymptotics.

3.2. Scheme of constructing the singular part of the asymptotics

Deducing the differential equation for the regular part $U_N(x,t,\varepsilon)$, at the same time we obtain the differential equation for the singular part $V_N(x,t,\tau,\varepsilon)$. Its splitting in a standard way allows us to come to a system of relations for the terms of the singular part of asymptotics (3) of the following form

$$\frac{\partial^3 V_0}{\partial \tau^3} + a_0(x,t)\frac{\partial V_0}{\partial \tau}\,\varphi'(t) - b_0(x,t)\left(u_0\frac{\partial V_0}{\partial \tau} + V_0\frac{\partial V_0}{\partial \tau}\right) = 0, \tag{10}$$

$$\frac{\partial^3 V_j}{\partial \tau^3} + a_0(x,t)\frac{\partial V_j}{\partial \tau}\,\varphi'(t) - b_0(x,t)\left(u_0\frac{\partial V_j}{\partial \tau} + \frac{\partial}{\partial \tau}\,(V_0 V_j)\right) = F_j(x,t,\tau), \quad j = \overline{1, 2N}, \tag{11}$$

where functions

$$F_j(x,t,\tau) = F_j(t, V_0(x,t,\tau), \ldots, V_{j-1}(x,t,\tau), u_0(x,t), \ldots, u_j(x,t))$$

are recurrently calculated with the functions $u_0(x,t)$, $u_1(x,t)$, ..., $u_j(x,t)$, $V_0(x,t,\tau)$, $V_1(x,t,\tau)$, ..., $V_{j-1}(x,t,\tau)$, $j = \overline{1,2N}$.

According to definition of the asymptotic step-like solution the functions $V_0(x,t,\tau) \in G_0$ and $V_j(x,t,\tau) \in G_1$, $j = \overline{1,2N}$. Taking into account these properties we solve the problem of determining the singular terms from equations (10), (11) in the following way:

1) firstly, we suppose that the function $\varphi(t)$ is known apriori and solve equations (10), (11) on the discontinuity curve $\Gamma = \{(x,t) \in \mathbf{R} \times [0;T] \colon x = \varphi(t)\}$;

2) secondly, we find the functions $V_0(x,t,\tau) \in G_0$, $V_j(x,t,\tau) \in G_1$, $j = \overline{1,2N}$, on the curve Γ. While solving the equation for these functions we find differential equation for the function $\varphi = \varphi(t)$;

3) on the next stage we prolong the functions $V_j(x,t,\tau)$, $j = \overline{0,2N}$, from the discontinuity curve Γ in such way that the found singular terms belong to the space G_0.

3.3. Determining the main term on the discontinuity curve

Let us consider the singular terms $V_j(x,t,\tau)$, $j = \overline{0,2N}$, on the curve Γ. Functions $v_j = v_j(t,\tau) = V_j(x,t,\tau)\big|_{x=\varphi(t)}$, $j = \overline{0,2N}$, satisfy partial differential equations:

$$\frac{\partial^3 v_0}{\partial \tau^3} + a_0(\varphi,t)\varphi'(t)\frac{\partial v_0}{\partial \tau} - b_0(\varphi,t)\left[u_0(\varphi,t)\frac{\partial v_0}{\partial \tau} + v_0\frac{\partial v_0}{\partial \tau}\right] = 0, \tag{12}$$

$$\frac{\partial^3 v_j}{\partial \tau^3} + a_0(\varphi,t)\varphi'(t)\frac{\partial v_j}{\partial \tau} - b_0(\varphi,t)\left[u_0(\varphi,t)\frac{\partial v_j}{\partial \tau} + \frac{\partial}{\partial \tau}(v_0 v_j)\right] = \mathcal{F}_j(t,\tau), \quad j = \overline{1,2N},$$

where values

$$\mathcal{F}_j(t,\tau) = F_j(t, V_0(x,t,\tau), \ldots, V_{j-1}(x,t,\tau), u_0(x,t), \ldots, u_j(x,t))\big|_{x=\varphi(t)}$$

are found recurrently after the functions $u_0(x,t)$, $u_1(x,t)$, ..., $u_j(x,t)$, $V_0(x,t,\tau)$, $V_1(x,t,\tau)$, ..., $V_{j-1}(x,t,\tau)$, $j = \overline{1,2N}$.

In particular, we have

$$\mathcal{F}_1(t,\tau) = a_0(\varphi,t)\frac{\partial v_0}{\partial t} + \left[-\tau a_{0x}(\varphi,t)\varphi' + b_0(\varphi,t)(\tau u_{0x}(\varphi,t) + u_1(\varphi,t)) + \right.$$
$$\left. + \tau b_{0x}(\varphi,t)(u_0(\varphi,t) + v_0)\right]\frac{\partial v_0}{\partial \tau} + b_0(\varphi,t)u_{0x}(\varphi,t)v_0, \quad \varphi = \varphi(t).$$

Move on analysis of equation (12) for main term of the singular part of the asymptotics. By integrating the equation with respect to the variable τ, we have:

$$\frac{d^2 v_0}{d\tau^2} = -a_0(\varphi,t)\varphi'(t)v_0(t,\tau) + b_0(\varphi,t)u_0(\varphi,t)v_0(t,\tau) + \frac{1}{2}b_0(\varphi,t)v_0^2(t,\tau) + c_1(t). \tag{13}$$

Due to the property $v_0(t,\tau) \in G_0$ we can put $c_1(t) \equiv 0$, Now multiplying equation (13) by $dv_0/d\tau$ and integrating with respect to τ we obtain equality:

$$\frac{d}{d\tau}\left(\frac{dv_0}{d\tau}\right)^2 = -\left[a_0(\varphi,t)\varphi'(t) - b_0(\varphi,t)u_0(\varphi,t)\right]\frac{dv_0^2}{d\tau} + \frac{1}{3}b_0(\varphi,t)\frac{dv_0^3}{d\tau},$$

or

$$\left(\frac{dv_0}{d\tau}\right)^2 = -a_0(\varphi,t)v_0^2\varphi'(t) + b_0(\varphi,t)u_0(\varphi,t)v_0^2 + \frac{1}{3}b_0(\varphi,t)v_0^3 + c_2(t).$$

From the property $v_0(t,\tau) \in G_0$ we deduce relation $c_2(t) \equiv 0$. So, the solution to equation (12) in the space G_0 is written as

$$v_0(t,\tau) = -3\,\frac{A(\varphi,\varphi',t)}{b_0(\varphi,t)}\,\cosh^{-2}\left(\frac{\sqrt{A(\varphi,\varphi',t)}}{2}(\tau + c_0)\right) \tag{14}$$

provided the condition

$$A(\varphi,\varphi',t) > 0, \tag{15}$$

where

$$A(\varphi,\varphi',t) = -a_0(\varphi,t)\varphi'(t) + b_0(\varphi,t)u_0(\varphi,t),$$

a function $\varphi = \varphi(t) \in C^{(1)}([0;T])$, $c_0 = c_0(t)$ is a constant of integrating.

Thus, we conclude the following lemma.

Lemma 3. *Under condition (15) a solution to equation (12) in the space G_0 is written in form (14), where $\varphi = \varphi(t) \in C^{(1)}([0;T])$ is an arbitrary function and c_0 is a constant.*

3.4. Accuracy of the main term of the asymptotic solution

While studying mathematical models with a small perturbation, much attention is paid to the main term of the asymptotic expansion and its accuracy.

Because $v_0(t,\tau) \in G_0$ we can put

$$V_0(x,t,\tau) = v_0(t,\tau). \tag{16}$$

The following result is true.

Theorem 1. *Let coefficients $a_0(x,t)$, $b_0(x,t) \in C^{(1)}(\mathbf{R}\times[0;T])$ be such that condition $a_0(x,t)b_0(x,t) \neq 0$ holds, $u_0(x,t) \in C^{\infty}(\mathbf{R}\times[0;T])$ be a solution of equation (4), $V_0(x,t,\tau)$ is defined by formulas (16), (14), the function $\varphi(t) \in C^{(1)}([0;T])$ satisfies inequality (15). Then the main term of asymptotics (3) is written as*

$$Y_0(x,t,\varepsilon) = u_0(x,t) + V_0(x,t,\tau), \quad \tau = \frac{x-\varphi(t)}{\sqrt{\varepsilon}}, \quad (x,t) \in \mathbf{R}\times[0;T]. \tag{17}$$

The function satisfies equation (2) with accuracy $\mathrm{O}(1)$. Moreover, it satisfies equation (2) with accuracy $\mathrm{O}(\sqrt{\varepsilon})$ as $\tau \to \pm\infty$ on the set $\mathbf{R}\times[0;T]$.

Proof. To prove the theorem we substitute expression (17) into (2) and consider residual function for the equation, i.e.

$$g_0(x,t,\varepsilon) = -\varepsilon\left(\frac{\partial^3 u_0}{\partial x^3} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\frac{\partial^3 V_0}{\partial \tau^3}\right) + a(x,t,\varepsilon)\left(\frac{\partial u_0}{\partial t} + \frac{\partial V_0}{\partial t} - \frac{1}{\sqrt{\varepsilon}}\varphi'(t)\,\frac{\partial V_0}{\partial \tau}\right) + b(x,t,\varepsilon)\,(u_0 + V_0)\left(\frac{\partial u_0}{\partial x} + \frac{1}{\sqrt{\varepsilon}}\frac{\partial V_0}{\partial \tau}\right). \tag{18}$$

Taking into account equations (4), (10), (12) we get the asymptotic relation

$$g_0(x,t,\varepsilon) = -\frac{1}{\sqrt{\varepsilon}}\,(a_0(x,t) - a_0(\varphi,t))\,\varphi'(t)\frac{\partial V_0}{\partial \tau} + \frac{1}{\sqrt{\varepsilon}}\,(b_0(x,t)u_0(x,t) - b_0(\varphi,t)u_0(\varphi,t))\,\frac{\partial V_0}{\partial \tau} + \frac{1}{\sqrt{\varepsilon}}\,(b_0(x,t) - b_0(\varphi,t))\,V_0\frac{\partial V_0}{\partial \tau} + \mathrm{O}(1). \tag{19}$$

Let us estimate every term in (19). Because of the property $V_0(x,t,\tau) = v_0(t,\tau) \in G_0$ the following inequalities are true:

$$\left|(a_0(x,t) - a_0(\varphi(t),t))\frac{\partial V_0}{\partial \tau}\right| \leqslant c_0|x-\varphi(t)| \left|\frac{\partial V_0}{\partial \tau}\right| \leqslant c_0\sqrt{\varepsilon}\tau \left|\frac{\partial V_0}{\partial \tau}\right| \leqslant \sqrt{\varepsilon}c_1,$$

where c_0, c_1 depend on a compact $K \subset \mathbf{R} \times [0;T]$.

Analogously we have

$$\left|(b_0(x,t) - b_0(\varphi(t),t))v_0\frac{\partial v_0}{\partial \tau}\right| \leqslant \sqrt{\varepsilon}c_2, \quad \left|(b_0(x,t)u_0(x,t) - b_0(\varphi(t),t)u_0(\varphi(t),t))v_0\frac{\partial v_0}{\partial \tau}\right| \leqslant \sqrt{\varepsilon}c_3,$$

where constants c_2, c_3 depend on a compact $K \subset \mathbf{R} \times [0;T]$.

Thus, for any compact $K \subset \mathbf{R} \times [0;T]$ the function $g_0(x,t,\varepsilon)$ is bounded for all $(x,t) \in K$.

Since $V_0(x,t,\tau) = v_0(t,\tau)$ is a quickly decreasing function, from (18), (19) we deduce that asymptotic solution (17) satisfies equation (2) with accuracy $\mathrm{O}(\sqrt{\varepsilon})$ as $\tau \to \pm\infty$ on the set $\mathbf{R} \times [0;T]$.

The theorem is proven. ∎

4. Conclusions

We apply the nonlinear WKB-technique and present the algorithm of constructing the asymptotic step-like solution to the singularly perturbed Korteweg–de Vries equation with variable coefficients in detail. The theorem on the accuracy of the main term of the asymptotic solution is given.

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Асимптотичний аналіз рівняння Кортевега–де Фріза за допомогою нелінійного методу ВКБ

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Дана стаття стосується побудови асимптотичних солітоноподібних розв'язків для рівняння Кортевега–де Фріза зі змінними коефіцієнтами та малим параметром при старшій похідній. Такі асимптотичні розв'язки вивчаються для широкого класу рівнянь з частинними похідними, які отримуються при математичному моделюванні процесів і явищ для випадку неоднорідних (за просторовою і часовою змінними) середовищ за наявності малої дисперсії і які є узагальненням певних інтегровних моделей. Шуканий розв'язок будується за допомогою нелінійного методу ВКБ, відповідно до якого асимптотичний розв'язок зображується у вигляді суми регулярної і сингулярної частин асимптотики. Якщо регулярна частина такого наближеного розв'язку математично описує фон, на якому відбувається хвильовий процес, то сингулярна частина цього розв'язку відображає характерні особливості, які пов'язані із солітонними властивостями рівняння Кортевега–де Фріза. Розглядається новий тип асимптотичних солітоноподібних розв'язків, коли головний доданок сингулярної частини шуканого асимптотичного розв'язку є швидко спадною функцією фазової змінної τ, а інші доданки є функціями сходинкового типу, тобто мають певну асимптотику на нескінченності. З огляду на ці властивості побудований асимптотичний розв'язок називається асимптотичним солітоноподібним розв'язком сходинкового типу. Представлено алгоритм побудови асимптотичних розв'язків даного типу, детально описано знаходження регулярної і сингулярної частин асимптотики, встановлено точність, з якою головний член побудованого асимптотичного розв'язку задовольняє вихідне рівняння.

Ключові слова: *рівняння Кортевега–де Фріза, нелінійний метод ВКБ, асимптотичний аналіз, сингулярне збурення, солітон, асимптотичний розв'язок.*