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Nonlinear method for determining external orientation elements of digital images obtained from drone

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We have suggested the method of application of a direct solving of the systems of nonlinear equations for finding the elements of external orientation (EEO) to perform aerial photography by unmanned aerial vehicles (UAVs).
The elements of external orientation functions search for the minimum of the function F, which is the sum of squares of coordinate differences on the image and is calculated by the measured coordinates on the ground, or the minimum of the function G, which is constructed using co-linearity and is the sum of squares of differences of the given coordinates X_i, Y_i, Z_i $(i = 1, 2, \ldots, n)$ on the ground and those which were calculated by the values x_i, y_i $(i = 1, 2, \ldots, n)$ measured on the image. In contrast to the classical approach, the choice of such a type of function is due to the possibility of implementing the algorithm using mathematical packages. Since some of the unknown coordinates X_i, Y_i, Z_i (the origin of the coordinate system is the center of projection) are included in the function G as arguments linearly, fulfillment of the conditions of the minimum of this function (equality of partial zero derivatives) in this case is simpler. This allows us to determine them through the angular elements of the EEO, which reduces the system of six equations to the system of three equations, being dependent on the angular elements. The function G is differentiated with respect to the variables dependent on the angular elements to obtain the three other equations. The obtained in this way system of equations is solved by the parameter variation method and gives us the solution of the required EEOs with a given accuracy.
The proposed algorithm gives us a real opportunity to clarify the values of EEO, moreover, the linear EEOs are determined with maximum accuracy, that makes it possible to increase the accuracy of the spatial coordinates of the points of the terrain.
The application of digital image processing from UAVs will significantly extend the range of implementation of aerial photography from UAVs to solve a variety of topographic, cadastral and engineering problems.
The proposed technique was tested on the relevant materials of aerial photography from UAVs at control points, which made it possible to confirm the optimality of the technique.

Keywords: *method of variation of parameters, unmanned aerial vehicle, elements of external orientation, nonlinear equations.*

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1. Introduction

The creation of topographic and cadastral plans of lands involves the usage of digital data obtained as a result of aerial photography, that requires the determination of the elements of external orientation (EEO) [1, 2]. Their application distinguishes the following technological problems in the use of unmanned aerial vehicles (UAVs) in topographic aerial photography: connection to the stabilization of the UAV during the flight and maintenance (maintaining a constant speed, flight and its straightness) and reduction of the inclination angles.

Failure to meet these conditions leads to errors in aerial photography materials, which is emphasized in [1, 3], where the ways of their possible elimination are also considered.

Mainly for UAVs, EEO values obtained by direct measurements are used as some preliminary approximation of parameters with their subsequent refinement by other methods, mainly based on minimizing the residual function [4,5]. Classically, this problem is solved by decomposing the residual function into a Taylor series according to the elements of EEO with their subsequent refinement: making corrections that are obtained from a system of linear equations. The iterative process ends if the parameters are the same with a given accuracy. Solving the problem, it is more logical to perform a check to achieve the minimum of the residual function or to obtain a solution from this condition directly. Moreover, available mathematical software allows this to realize [4,6]. However, the usage of these mathematical packages is associated with some peculiarities. First of all, a good initial approximation is required to ensure the convergence of the methods used, and the result of the execution can give us a local minimum, which cannot be classified as a solution. This raises the question of reducing the problem to a system of nonlinear equations to obtain a solution by approximate methods and compare it with obtained solution using mathematical packages. The parameter variation's method is preferred, which does not require checking the conditions of convergence, but is a certain analogue of the method of half division for one variable [7,8].

This issue is also studied in military. Possibilities of using UAVs for military actions [9], reconnaissance of the area are covered in the articles [10,11]. In order to improve the determination of the coordinates of unmanned aerial vehicles in the area of the anti-terrorist operation [12], the authors assessed the accuracy of determining the location of radio emitting targets by difference-range method in a mobile passive radar system based on short-range anti-aircraft systems [13]. The results of the study show that the errors in measuring coordinates of the difference-range method are insignificant and at some positions of the UAV are important compared to its size. The obtained dependencies allow us to choose the optimal, in terms of minimal errors in the coordinates of the UAV, location of combat vehicles from passive direction finders.

2. Statement of the problem

One of the main steps in determining the coordinates of points on the ground is to obtain elements of external orientation. The accuracy of their calculation by hardware is low due to the impossibility of placing bulky equipment on the UAV. The main way to find exact values of obtained elements is to solve the inverted photogrammetric resection in which the coordinates of reference points appear [14,15]. Today, the generally accepted method is based on the decomposition of formulas for determining the coordinates on the ground for the elements of the EEO with their subsequent refinement. It is implemented in most software packages that solve the problem of calculating coordinates. However, it is not possible to verify the correctness of this definition, because direct verification of the found parameters is not performed. The only criterion of reliability is the visual compatibility of the results. Therefore, it is not necessary to raise the issue of improving the calculation results, because the tools for influencing accuracy are not known.

For better understanding, there is a question of clear mathematical formulation of the problem: we are to define such values α, ω, κ (angular elements of external orientation), and the center of projection X_S, Y_S, Z_S, which minimize the target function F as follows

$$F(\alpha,\omega,\kappa,X_S,Y_S,Z_S)=\sum_{i=1}^{n}\left(x_i-x_0+f\frac{a_1(X_i-X_S)+a_2(Y_i-Y_S)+a_3(Z_i-Z_S)}{c_1(X_i-X_S)+c_2(Y_i-Y_S)+c_3(Z_i-Z_S)}\right)^2 + \left(y_i-y_0+f\frac{a_1(X_i-X_S)+a_2(Y_i-Y_S)+a_3(Z_i-Z_S)}{c_1(X_i-X_S)+c_2(Y_i-Y_S)+c_3(Z_i-Z_S)}\right)^2, \quad (1)$$

where

$$a_1=\cos\alpha\cos\kappa-\sin\alpha\sin\omega\sin\kappa,\quad a_2=-\cos\alpha\sin\kappa-\sin\alpha\sin\omega\cos\kappa,\quad a_3=-\sin\alpha\cos\omega,$$
$$b_1=\cos\omega\sin\kappa,\quad b_2=\cos\omega\cos\kappa,\quad b_3=-\sin\omega,$$
$$c_1=\sin\alpha\cos\kappa+\cos\alpha\sin\omega\sin\kappa,\quad c_2=-\sin\alpha\sin\kappa+\cos\alpha\sin\omega\cos\kappa,\quad c_3=\cos\alpha\cos\omega,$$

f is the focal length of the digital camera, x_0, y_0 are planned elements of internal orientation, x_i, y_i are coordinates in the image and X_i, Y_i, Z_i are coordinates in the area $0 \leqslant i \leqslant n$, where n is number of points (reference points).

It should be noted that the following approach is usually used in the classical scheme: the minimum of the function is found for the linearized function F. Also, the optimality test is carried out not by direct substitution in equation (1), but by comparing the previous and subsequent values of parameters in the iterative process [14]. Considering that the coordinates in the image and the field are in the relationship

$$X_i - X_S = (Z_i - Z_s)\,\frac{a_1(x_i - x_0) + a_2(y_i - y_0) + a_3 f}{c_1(x_i - x_0) + c_2(y_i - y_0) + c_3 f},$$
$$Y_i - Y_S = (Z_i - Z_s)\,\frac{b_1(x_i - x_0) + b_2(y_i - y_0) + b_3 f}{c_1(x_i - x_0) + c_2(y_i - y_0) + c_3 f},$$

we can conclude that

$$G(\alpha, \omega, \kappa, X_S, Y_S, Z_S) = \sum_{i=1}^{n} \left(X_i - X_S - (Z_i - Z_S)\,\overline{x}_i\right)^2 + \left(Y_i - Y_S - (Z_i - Z_S)\,\overline{y}_i\right)^2, \qquad (2)$$

where

$$\overline{x}_i(\alpha, \omega, \kappa) = \frac{a_1 x_i + a_2 y_i - a_3 f}{c_1 x_i + c_2 y_i - c_3 f}, \qquad \overline{y}_i(\alpha, \omega, \kappa) = \frac{b_1 x_i + b_2 y_i - b_3 f}{c_1 x_i + c_2 y_i - c_3 f}. \qquad (3)$$

Substitution of EEO parameters found with the help of mathematical software in equations (1), (2) does not give a minimum value. Using the mathematical packages [7], the parameters of EEO can be specified, and as a result the value of a minimum can be significantly improved. However, they cannot be considered as final ones. After all, the condition for the existence of an extremum of a function of many variables is the equality to zero of its partial derivatives, and it is not satisfied even for refined values. Therefore, the algorithm for finding EEO needs to be significantly adjusted: we can treat EEO as a solution for a system of nonlinear equations. Let us focus on the choice of the objective function, F or G. Optimization can be performed by the function F that is essentially another record of relation (2). The results of calculations for these two objective functions are similar, but do not coincide. This can be explained by the instability of the solution (the problem is incorrect). Therefore, the choice of the type of function (1) or (2) is determined by the task. If there is a question of determining the coordinates of points on the ground, it is advisable to choose the condition (2); in other cases, it is possible to use (1).

3. Presenting main results

The function G of the elements of external orientation (2), for which the minimum is searched, is simpler than expression (1) for the function F, because the elements X_S, Y_S, Z_S are included in the function as arguments in a linear manner. Therefore, the system of equations obtained by differentiating the function G

$$\begin{cases} \dfrac{\partial G}{\partial X_S} = 0, \quad \dfrac{\partial G}{\partial Y_S} = 0, \quad \dfrac{\partial G}{\partial Z_S} = 0, \\ \dfrac{\partial G}{\partial \alpha} = 0, \quad \dfrac{\partial G}{\partial \omega} = 0, \quad \dfrac{\partial G}{\partial \kappa} = 0 \end{cases} \qquad (4)$$

can be reduced to the system of three equations, depending on the angular elements. Obtained system of equations is solved by the method of parameter variation represented in [16].

It should be noted that the system of equations (3) can be solved using the mathematical packages involving differentiation [7]. However, there are some difficulties, because conditions (3) also determine the local extreme values. Thus, there is a need of finding solutions that are complement or alternative to the above approach. To do this, choose the form of the system (3) for the function G (2). Differentiation with respect to the variables X_S, Y_S, Z_S gives us

$$
\begin{aligned}
&nX_S - Z_S \sum_{i=1}^{n} \overline{x}_i = \sum_{i=1}^{n} \overline{X}_i - \sum_{i=1}^{n} Z_i \overline{x}_i,\\
&nY_S - Z_S \sum_{i=1}^{n} \overline{y}_i = \sum_{i=1}^{n} \overline{Y}_i - \sum_{i=1}^{n} Z_i \overline{y}_i,\\
&X_S \sum_{i=1}^{n} \overline{x}_i + Y_S \sum_{i=1}^{n} \overline{y}_i - X_S \sum_{i=1}^{n} \left(\overline{x}_i^2 + \overline{y}_i^2\right) = \sum_{i=1}^{n} \left(X_i \overline{x}_i + Y_i \overline{y}_i\right) - \sum_{i=1}^{n} Z_i \left(\overline{x}_i^2 + \overline{y}_i^2\right).
\end{aligned} \tag{5}
$$

We can obtain the rest of the equations in the following way. First, differentiate the function G with respect to the additional variables – the components of the vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$. Taking into account the orthogonality of the basis yields the conditional extremum problem, the Lagrange function of which is determined as

$$
\begin{aligned}
\psi(a, b, c, \lambda) = \sum_{i=1}^{n} (u_i - w_i \overline{x}_i)^2 + (v_i - w_i \overline{y}_i)^2 + \lambda_1 (a_1^2 + a_2^2 + a_3^2 - 1) + \lambda_2 (b_1^2 + b_2^2 + b_3^2 - 1)\\
+ \lambda_3 (c_1^2 + c_2^2 + c_3^2 - 1) + \lambda_4 (a_1 b_1 + a_2 b_2 + a_3 b_3) + \lambda_5 (a_1 c_1 + a_2 c_2 + a_3 c_3) + \lambda_6 (b_1 c_1 + b_2 c_2 + b_3 c_3),
\end{aligned} \tag{6}
$$

where $u_i = X_i - X_S$, $v_i = Y_i - Y_S$, $w_i = Z_i - Z_S$, and $\lambda = \lambda(\lambda_1, \lambda_2, \ldots, \lambda_6)$ are Lagrange factors.

Differentiate the expression (6) with respect to the arguments a_1, a_2, a_3 taking into account the conditions of the minimum:

$$
\begin{aligned}
\frac{\partial \psi}{\partial a_1} &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{x}_i) \frac{w_i x_i}{\overline{z}_i} + 2\lambda_1 a_1 + \lambda_4 b_1 + \lambda_5 c_1 = 0,\\
\frac{\partial \psi}{\partial a_2} &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{x}_i) \frac{w_i x_i}{\overline{z}_i} + 2\lambda_1 a_2 + \lambda_4 b_2 + \lambda_5 c_2 = 0,\\
\frac{\partial \psi}{\partial a_3} &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{x}_i) \frac{w_i (-f)}{\overline{z}_i} + 2\lambda_1 a_3 + \lambda_4 b_3 + \lambda_5 c_3 = 0.
\end{aligned}
$$

Transforming the last three equations and bearing in mind the orthogonality, we obtain

$$
\begin{aligned}
\frac{\partial \psi}{\partial a_1} a_1 + \frac{\partial \psi}{\partial a_2} a_2 + \frac{\partial \psi}{\partial a_3} a_3 &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{x}_i) w_i \overline{x}_i + 2\lambda_1 = 0,\\
\frac{\partial \psi}{\partial a_1} b_1 + \frac{\partial \psi}{\partial a_2} b_2 + \frac{\partial \psi}{\partial a_3} b_3 &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{y}_i) w_i \overline{y}_i + \lambda_4 = 0,\\
\frac{\partial \psi}{\partial a_1} c_1 + \frac{\partial \psi}{\partial a_2} c_2 + \frac{\partial \psi}{\partial a_3} c_3 &= -2 \sum_{i=1}^{n} (u_i - w_i \overline{x}_i) w_i + \lambda_5 = 0.
\end{aligned} \tag{7}
$$

Similarly, differentiating with respect to the variables b_1, b_2, b_3, we obtain

$$
\begin{aligned}
\frac{\partial \psi}{\partial b_1} b_1 + \frac{\partial \psi}{\partial b_2} b_2 + \frac{\partial \psi}{\partial b_3} b_3 &= -2 \sum_{i=1}^{n} (v_i - w_i \overline{y}_i) w_i \overline{y}_i + 2\lambda_2 = 0,\\
\frac{\partial \psi}{\partial b_1} a_1 + \frac{\partial \psi}{\partial b_2} a_2 + \frac{\partial \psi}{\partial b_3} a_3 &= -2 \sum_{i=1}^{n} (v_i - w_i \overline{y}_i) w_i \overline{x}_i + \lambda_4 = 0,\\
\frac{\partial \psi}{\partial b_1} c_1 + \frac{\partial \psi}{\partial b_2} c_2 + \frac{\partial \psi}{\partial b_3} c_3 &= -2 \sum_{i=1}^{n} (v_i - w_i \overline{y}_i) w_i + \lambda_6 = 0.
\end{aligned} \tag{8}
$$

Now, differentiate with respect to the variables c_1, c_2, c_3; it gives us

$$\frac{\partial\psi}{\partial c_1} = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)\frac{w_i\overline{x}_i x_i}{\overline{z}_i^2} + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)\frac{w_i\overline{y}_i x_i}{\overline{z}_i} + 2\lambda_3 c_1 + \lambda_5 a_1 + \lambda_6 b_1 = 0,$$
$$\frac{\partial\psi}{\partial c_2} = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)\frac{w_i\overline{x}_i y_i}{\overline{z}_i^2} + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)\frac{w_i\overline{y}_i y_i}{\overline{z}_i} + 2\lambda_3 c_2 + \lambda_5 a_2 + \lambda_6 b_2 = 0,$$
$$\frac{\partial\psi}{\partial c_3} = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)\frac{w_i\overline{x}_i(-f)}{\overline{z}_i^2} + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)\frac{w_i\overline{y}_i(-f)}{\overline{z}_i} + 2\lambda_3 c_3 + \lambda_5 a_3 + \lambda_6 b_3 = 0. \quad (9)$$

After these transformations, we obtain

$$\frac{\partial\psi}{\partial c_1}a_1 + \frac{\partial\psi}{\partial c_2}a_2 + \frac{\partial\psi}{\partial c_3}a_3 = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{x}_i^2 + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{y}_i\overline{x}_i + \lambda_5 = 0,$$
$$\frac{\partial\psi}{\partial c_1}b_1 + \frac{\partial\psi}{\partial c_2}b_2 + \frac{\partial\psi}{\partial b_3}b_3 = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{y}_i\overline{x}_i + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{y}_i^2 + \lambda_6 = 0,$$
$$\frac{\partial\psi}{\partial c_1}c_1 + \frac{\partial\psi}{\partial c_2}c_2 + \frac{\partial\psi}{\partial b_3}c_3 = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{x}_i + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{y}_i + \lambda_4 = 0. \quad (10)$$

In this set of equations, we exclude the variables λ_4, λ_5, λ_6 and as a result we obtain

$$2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{x}_i = 2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{y}_i,$$
$$2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{x}_i^2 + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{y}_i\overline{x}_i = -2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i,$$
$$2\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{y}_i\overline{x}_i + 2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i\overline{y}_i^2 = -2\sum_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i.$$

The set of these equations (5) together with (2) give us the following system

$$\begin{cases} nX_S - Z_S\sum\limits_{i=1}^{n}\overline{x}_i = \sum\limits_{i=1}^{n}X_i - \sum\limits_{i=1}^{n}Z_i\overline{x}_i, \\ nY_S - Z_S\sum\limits_{i=1}^{n}\overline{y}_i = \sum\limits_{i=1}^{n}Y_i - \sum\limits_{i=1}^{n}Z_i\overline{y}_i, \\ X_S\sum\limits_{i=1}^{n}\overline{x}_i + Y_S\sum\limits_{i=1}^{n}\overline{y}_i - Z_S\sum\limits_{i=1}^{n}\left(\overline{x}_i^2 + \overline{y}_i^2\right) = \sum\limits_{i=1}^{n}(X_i\overline{x}_i + Y_i\overline{y}_i) - \sum\limits_{i=1}^{n}Z_i\left(\overline{x}_i^2 + \overline{y}_i^2\right), \\ \sum\limits_{i=1}^{n}v_i w_i\overline{x}_i - \sum\limits_{i=1}^{n}u_i w_i\overline{y}_i = 0, \\ \sum\limits_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{x}_i^2 + \sum\limits_{i=1}^{n}(u_i - w_i\overline{y}_i)w_i\overline{y}_i\overline{x}_i + \sum\limits_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i = 0, \\ \sum\limits_{i=1}^{n}(u_i - w_i\overline{x}_i)w_i\overline{y}_i\overline{x}_i + \sum\limits_{i=1}^{n}(u_i - w_i\overline{y}_i)w_i\overline{y}_i^2 + \sum\limits_{i=1}^{n}(v_i - w_i\overline{y}_i)w_i = 0. \end{cases} \quad (11)$$

The equation (11) is used to estimate the accuracy of determining the elements of the EEO, as they reflect the approximation of the function G to the optimal value or deviation from it. In addition, relation (11) can be interpreted as a nonlinear system of six equations with unknown variables α, ω, κ, X_S, Y_S, Z_S. This system of equations can be reduced to a system of three equations with unknown variables α, ω, κ preexpressing the variables X_S, Y_S, Z_S from the first three systems (11):

$$\begin{cases} G_1(\alpha,\omega,\kappa) = 0, \\ G_2(\alpha,\omega,\kappa) = 0, \\ G_3(\alpha,\omega,\kappa) = 0. \end{cases} \quad (12)$$

To find the solution (12), we use the method of variation of parameters [16]. The sense of this method is in searching a solution, determined by the following scheme.

1. Fix one of the unknown variables, for example, $\alpha = p$;

2. Solve a system of two equations

$$\begin{cases} G_2(p, \omega, \kappa) = 0, \\ G_3(p, \omega, \kappa) = 0, \end{cases} \tag{13}$$

for which we again apply the method of variations, choosing the parameter, $\omega = p_1$. As a result, we obtain two equations with one variable κ

$$G_2(p, p_1, \kappa) = 0, \tag{14}$$

$$G_3(p, p_1, \kappa) = 0. \tag{15}$$

If there is a solution, we find it for two equations (14), (15): $\kappa_1(p, p_1)$, $\kappa_2(p, p_1)$ by the method of half division, which provides the localization of the solution (if it exists) and its refinement by reducing the length of the interval of its location.

3. We make the difference between their root values

$$\delta(p, p_1) = \kappa_1(p, p_1) - \kappa_2(p, p_1) = 0 \tag{16}$$

and solve the equation with respect to p_1, by the method of half division. As a result, we obtain the parameter $p_1(p)$ for which equations (14) and (15) hold. Therefore,

$$\omega^{(i)}(p), \quad \kappa^{(i)}(p), \quad i = 0, 1, \dots \tag{17}$$

is the solution of system (13) for a fixed p.

4. Substitute the value (17) in the first equation of the system (12). We obtain another equation with one variable p

$$G_1\big(\alpha, \omega^{(i)}(p), \kappa^{(i)}(p)\big) = 0, \tag{18}$$

which determines $\alpha^{(i)}(p)$.

5. Now, consider the difference $\delta_1(p) = p - \alpha^{(i)}(p)$ and solve the equation with respect to p

$$\delta_1(p) = p - \alpha^{(i)}(p) = 0 \tag{19}$$

by changing the parameter p (for example, increasing with a certain step) and repeating the points 1–6 until condition (19) is met with a given accuracy, which is provided by changing the sign of the function $\delta_1(p)$ in two adjacent values of p. Each step is an iteration with the number i.

6. When the condition (19) is satisfied, we obtain the fact: the following values of parameters are determined that satisfy the equation (18) as well as the system (13) because it holds for all p, hence, for $p = \alpha^{(i)}(p)$.

The specifics of calculation of the values included in the functions G_1, G_2, G_3 require some clarification. This, primarily, concerns the accuracy of determining the linear and angular parameters for which these requirements are different. For example, for linear elements, the order of accuracy to centimeters is sufficient, and for angular to a second. Therefore, we divide the definition of these parameters into separate stages as follows: for sufficiently close to the optimal values of the parameters X_S, Y_S, Z_S at points 1–6, we find the angular elements α, ω, κ. For them, we find new values of X_S, Y_S, Z_S from the first three equations of system (11) and repeat steps 1–6. The process is performed until the required accuracy for linear elements is achieved. For angular EEO in the course of performance of points 1–6, the above accuracy of their definition is reached.

4. Analysis of the features of this technique on a specific example

Assume we have the coordinates on the ground and their values on the picture (Table 1). They can be used to calculate the EEO, which are presented in the first line of Table 2. To assess the accuracy of determining the EEO, perform the inverse transition, i.e., find the difference between the calculated and given coordinates on the ground and calculate their weighted average deviation by the formula:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(u_i - w_i\overline{x}_i)^2 + (v_i - w_i\overline{y}_i)^2}{2n}}.$$

As can be seen from Table 2 (1st row, 8th column), this value is quite significant, so there is a need to clarify the EEO. According to the last column of this table, the value of s is significantly smaller for the lines II and III, which indicates the specification of parameters whose values do not actually differ from the initial ones (for example, for angular elements no more than 2–3 seconds, which is within tolerance and more significant for linear ones [8]).

Table 1. Coordinates of reference points in the field and their coordinates on the image.

No	X (m)	Y (m)	Z (m)	x_i (mm)	y_i (mm)
1	5455868.561	670717.53	574.4528	10.666	14.473
2	5455710.546	670755.539	581.1358	10.548	−11.845
3	5455757.676	670693.335	576.7235	2.599	−1.760
4	5455834.278	670653.263	574.0395	−0.642	12.248
5	5455702.22	670689.28	584.143	−0.2962	−10.931
6	5455819.224	670627.874	573.773	−5.392	10.974
7	5455762.244	670673.223	575.9528	−0.445	−0.1824
8	5455771.558	670673.121	575.5768	−0.078	1.352
9	5455769.062	670691.815	576.0705	2.818	0.142

Table 2. Elements of the external orientation of the image.

No	α (degree)	ω (degree)	κ (degree)	X_S (m)	Y_S (m)	Z_S (m)	S (m)
I	$5°22'40.5''$	$0°54'6.45''$	$14°44'20.18''$	670650.1	5455759.04	784.62	9.26
II	$5°22'40.75''$	$0°54'4.45''$	$14°44'44.03''$	670655.844461	5455760.61351	785.92039661	0.2
III	$6°06'11.16''$	$1°22'14.83''$	$14°39'27.71''$	670653.215757	5455758.86862	784.98761149	0.12

However, for these two options (I, II lines of Table 2), the values of the parameters do not satisfy the conditions of equations (4) or (11) (Table 3, I, II lines), because these values are not minimal. Therefore, to find the optimal values, we apply the conditions (4). The results of the calculations are presented in the third line of Table 3. Analysis of the values allows us to conclude that the optimal results can be considered as the data of the third line, all values of which are actually zero. It should be noted that the EEO are significantly different from those originally defined. Therefore, there is a need for alternative verification of the results.

Table 3. The values of the derivatives of the function F by the EEO (for the values presented in Table 1).

No	$\frac{\partial G}{\partial \alpha}$	$\frac{\partial G}{\partial \omega}$	$\frac{\partial G}{\partial \kappa}$	$\frac{\partial G}{\partial X_S}$	$\frac{\partial G}{\partial Y_S}$	$\frac{\partial G}{\partial Z_S}$
I	53.355	15.052	−9.991	−482.036	−1.167E + 4	−3.346E + 3
II	−4.998E − 5	3.507E − 5	3.429E − 4	−53.162	−16.0314	−28.743
III	3.313E − 10	−1.563E − 9	5.354E − 11	−4.902E − 8	−7.809E − 8	2.878E − 7

To do this, use the algorithm proposed in the work. The initial data of linear elements are selected from Table 2, line III. Based on numerical experiments, the lower limits of the angles of external orientation are established, and for the parameter α, the range of change should be insignificant $h_\alpha = 0.001^\circ$ and for the other two elements it can vary in a wider range from -20° with step $h_{\omega,\kappa} = 5^\circ$.

In Table 4, line I shows the calculations of the initial values, line II represents the results of the algorithm. The analysis of the obtained results shows us that the values of the EEO elements obtained in two different ways give almost the same values. In this case, the angular elements coincide to the sixth decimal place (0.0001 sec), and linear to the fifth one (0.01 mm). This indicates the high accuracy of determining the elements of the proposed method. In practice, the accuracy of determining the spatial coordinates of terrain points is regulated to the second for angular elements and to the centimeter for linear elements; and these extra digits after the decimal point illustrate the convergence of the process.

Table 4. The results of the values of EEO, respectively, the initial and the calculated using proposed algorithm.

No	α (degree)	ω (degree)	κ (degree)	X_S (m)	Y_S (m)	Z_S (m)
I	$6^\circ 06' 11.16154908''$	$1^\circ 22' 14.83492794''$	$14^\circ 39' 27.717336768''$	670653.215757	5455758.86862	784.987611
II	$6^\circ 06' 11.16159520''$	$1^\circ 22' 14.83729''$	$14^\circ 39' 46.195737577''$	670653.215757	5455758.86862	784.987608

Some warnings should be made regarding to the implementation of the algorithm. Since one of the conditions when looking for solutions is to change the sign of the function, it is necessary to limit the narrow intervals of change of these parameters and take them close to the solution. These values can be obtained using other tools, such as software packages Digital, Mathcad. This process is allowed to be carried out by numerical experiments. For example, the possibility of a significant deviation from the solution of the elements ω, κ when performing the algorithm and the need for a more accurate approximation for $\alpha \approx (0.01^\circ)$ is established. The same refers to linear elements. We take them as defined by other methods, and their further refinement is implemented using the proposed algorithm.

When solving a system of linear equations by the method of variation of constants, a number of issues arise related to its implementation, and above all the possibility of obtaining a solution and the possibility of replacing it with other simpler methods of understanding. However, given the specifics of the problem, it is not possible to do so yet, because the behavior of the minimizing function is unpredictable.

Therefore, there is a need for further improvement of the algorithm using other techniques for solving equations, for example, using the method of chords and tangents to find the roots of a function of one variable [8]. This can be the subject of further research.

5. Conclusions

1. The accuracy of determining the EEOs defined by traditional methods is insufficient to find the coordinates on the ground.
2. Attracting additional conditions for the existence of the extremum clarifies the parameters of the external orientation of the image.
3. The use of the nonlinear method complements the solution of systems of nonlinear equations with the help of mathematical packages and helps to establish the limits of changes in the elements of external orientation.
4. In the future, it is planned to create approximate numerical methods for solving the system of nonlinear equations, which ensures the convergence of iterative processes to clarify the elements of the EEO digital images obtained by UAVs.

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Нелінійний метод визначення елементів зовнішнього орієнтування цифрових зображень отриманих з БПЛА

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Запропоновано метод застосування прямого розв'язування систем нелінійних рівнянь для знаходження елементів зовнішнього орієнтування (ЕЗО) для виконання аерофотознімань безпілотними літальними апаратами (БПЛА).
Елементи функцій зовнішньої орієнтації шукають як мінімум функції F, яка є сумою квадратів різниць координат на знімку та обчислених за виміряними координатами на місцевості або ж функції G, яка будується з використанням умов колінеарності та як сума квадратів різниць заданих координат X_i, Y_i, Z_i $(i = 1, 2, \ldots, n)$ на місцевості та обчислених за виміряними на знімку значеннями x_i, y_i $(i = 1, 2, \ldots, n)$. Вибір такого виду функції, на відміну від класичного підходу, обумовлений насамперед можливістю реалізації алгоритму за допомогою математичних пакетів. Виконання умов мінімуму функції G (рівність часткових похідних, які дорівнюють нулю) у цьому випадку є простішим, оскільки частина невідомих X_i, Y_i, Z_i (початок системи координат — центр проекції) входить у функцію як аргументи, лінійним чином. Це дає можливість визначити їх через кутові елементи зовнішнього орієнтування, що спрощує систему з шести рівнянь до системи трьох рівнянь, залежних від кутових елементів. Щоб отримати інші три рівняння, функцію G диференціюють за змінними, які залежать від кутових елементів. Отримана таким чином система рівнянь розв'язується методом варіації параметрів і дає розв'язок шуканих елементів зовнішнього орієнтування з відповідним початковим наближенням.
Запропонований алгоритм дає реальну можливість уточнити значення елементів зовнішнього орієнтування, до того ж лінійні елементи зовнішнього орієнтування визначаються з максимальною точністю, що дає можливість збільшити точність просторових координат точок місцевості.
Застосування цифрового опрацювання зображень з БПЛА дозволить значно розширити діапазон реалізації аерофотознімань з БПЛА для вирішення різноманітних топографічних, кадастрових та інженерних проблем.
Запропоновану методику апробовано на відповідних матеріалах аерофотознімань з БПЛА на контрольних пунктах, що підтвердило оптимальність методики.

Ключові слова: *метод варіації параметрів, безпілотний літальний апарат, елементи зовнішнього орієнтування, нелінійні рівняння.*