MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 3, pp. 471–482 (2022)

Legendre–Kantorovich method for Fredholm integral equations of the second kind

Arrai M.[1], Allouch C.[1], Bouda H.[1], Tahrichi M.[2]

[1] *University Mohammed I, Team MSC, FPN, LAMAO Laboratory, Nador, Morocco*
[2] *University Mohammed I, Team ANAA, EST, LANO Laboratory, Oujda, Morocco*

(Received 11 December 2021; Accepted 4 March 2022)

In the present paper, we consider polynomially based Kantorovich method for the numerical solution of Fredholm integral equation of the second kind with a smooth kernel. The used projection is either the orthogonal projection or an interpolatory projection using Legendre polynomial bases. The order of convergence of the proposed method and those of superconvergence of the iterated versions are established. We show that these orders of convergence are valid in the corresponding discrete methods obtained by replacing the integration by a quadrature rule. Numerical examples are given to illustrate the theoretical estimates.

Keywords: *Fredholm integral equation, projection operator, Legendre polynomial, superconvergence, quadrature rule, discrete method.*

2010 MSC: 35A01, 65L10, 65L12, 65L20, 65L70 **DOI:** 10.23939/mmc2022.03.471

1. Introduction

Consider the Fredholm integral equation defined on $\mathscr{C}[-1,1]$ by

$$u(s) - \int_{-1}^{1} \kappa(s,t)\, u(t)\, dt = f(s), \quad -1 \leqslant s \leqslant 1, \tag{1}$$

where κ is a smooth kernel, f is a real continuous function and u denotes the unknown function. The projection, degenerate kernel and Nyström methods are standard methods for finding numerical solutions of equations of type (1) (see [1–3]). It is well known that to get better precision in these methods in the case of piecewise polynomial approximation, the number of partition points should be increased. Hence in such cases, we should solve a large system of linear equations, which is computationally expensive.

The aim of this paper is to study the Kantorovich method and its discretized version to solve equation (1) using global polynomial basis functions rather than piecewise polynomial basis functions which reduces highly the size of linear system. In particular, Legendre polynomials can be used as basis functions which have nice property of orthogonality and low computational cost.

In a number of recent papers, various polynomially based numerical methods for linear integral equations were studied. The discrete Galerkin method using Legendre polynomials was introduced in Golberg [4] and its iterated version was proposed in Kulkarni and Gnaneshwar [5]. The convergence of the Legendre–Galerkin solution in the case of weakly singular kernels was considered in Panigrahi and Gnaneshwar [6]. Moreover, the Legendre multi-projection as well as its iterated version were studied in [7]. Other important results on the numerical solutions of nonlinear integral equations using Legendre polynomials can be found in [8–10].

Now for a summary of the paper. In Section 2, the proposed method is defined and the linear systems which need to be solved to obtain the approximate solutions are discussed. In Section 3, the orders of convergence of the proposed method and its iterated version for both the orthogonal projection and the interpolatory projection are obtained. In Section 4 we show that these orders of convergence are preserved after taking into account the errors introduced by the numerical quadrature rule. Numerical results are given in Section 5.

2. Method and notations

Let $\mathbb{X}_n$ denote the space of all polynomials of degree $\leqslant n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the Legendre polynomials $\{L_0, L_1, L_2, \dots, L_n\}$ defined by

$$L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1,1],$$
$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1, 2, \dots, n-1$$

form an orthogonal basis for $\mathbb{X}_n$. Since

$$\langle L_i, L_j \rangle = \begin{cases} \frac{2}{2i+1}, & i = j, \\ 0, & i \neq j, \end{cases}$$

then, an orthonormal basis for $\mathbb{X}_n$ is given by

$$\left\{ \varphi_i(s) = \sqrt{\tfrac{2i+1}{2}} L_i(s) \colon i = 0, 1, \dots n \right\}.$$

We use two types of projections from $\mathscr{C}[-1,1]$ to $\mathbb{X}_n$.

Orthogonal projection. For $u, v \in \mathscr{C}[-1,1]$, the inner product is given by

$$\langle u, v \rangle = \int_{-1}^{1} u(t)\, v(t)\, dt \quad \text{and norm is} \quad \|u\|_{L^2} = \left(\int_{-1}^{1} u(t)^2 dt \right)^{\frac{1}{2}}.$$

Let $\pi_n^G u$ be the orthogonal projection operator defined from $\mathscr{C}[-1,1]$ to $\mathbb{X}_n$. Then for all $u \in \mathscr{C}[-1,1]$, we have

$$\begin{aligned} \big(\pi_n^G u\big)(s) &= \sum_{i=0}^{n} \langle u, \varphi_i \rangle \varphi_i(s), \\ \big\langle \pi_n^G u, \varphi_i \big\rangle &= \langle u, \varphi_i \rangle, \quad i = 0, 1, \dots, n. \end{aligned} \tag{2}$$

Interpolatory projection. For $u \in \mathscr{C}[-1,1]$, let $\pi_n^C u$ denote the unique polynomial of degree n satisfying

$$\big(\pi_n^C u\big)(\tau_i) = u(\tau_i), \quad i = 0, 1, \dots, n, \tag{3}$$

where $\{\tau_0, \tau_1, \dots, \tau_n\}$ are zeros of the Legendre polynomial L_{n+1}. In the Lagrange form, $\pi_n^C u$ is

$$\big(\pi_n^C u\big)(s) = \sum_{j=0}^{n} u(\tau_j) \ell_j(s), \quad s \in [-1,1],$$

where ℓ_j is the unique polynomial of degree n that satisfies $\ell_j(\tau_i) = \delta_{ij}$. Clearly, π_n^C is a linear operator on $\mathscr{C}[-1,1]$, with the property $(\pi_n^C)^2 = \pi_n^C$. It is therefore a projection, having as range the set $\mathbb{X}_n$. Henceforth, we write π_n^G or π_n^C as π_n and for the rest of the paper we assume that $r \geqslant 1$. The crucial properties of π_n are given in the following lemma.

Lemma 1 (Golberg and Chen [11]). *Let $\pi_n \colon \mathscr{C}[-1,1] \to \mathbb{X}_n$ be the orthogonal or interpolatory projection operator defined by (2) and (3). There exists a constant $p > 0$ independent of n such that for $n \in \mathbb{N}$ and $u \in \mathscr{C}[-1,1]$,*

$$\|\pi_n u\|_{L^2} \leqslant p \|u\|_{L^2}, \tag{4}$$

$$\|u - \pi_n u\|_{L^2} \leqslant (1+p) \inf_{\phi \in \mathbb{X}_n} \|u - \phi\|_{L^2}. \tag{5}$$

Moreover, for any $u \in \mathscr{C}^r[-1,1]$,

$$\|u - \pi_n u\|_{L^2} \leqslant c_1 n^{-r} \|u^{(r)}\|_{L^2}, \tag{6}$$

$$\|u - \pi_n u\|_{\infty} \leqslant c_1 n^{\beta - r} \|u^{(r)}\|_{\infty}, \tag{7}$$

where c_1 is a constant independent of n, $\beta = \frac{3}{4}$ for the orthogonal projection and $\beta = \frac{1}{2}$ for the interpolatory projection.

The estimate (5) shows that $\|u - \pi_n u\|_{L^2} \to 0$ as $n \to \infty$ for all $u \in \mathscr{C}[-1,1]$, whereas the estimate (7) imply that $\|u - \pi_n u\|_\infty \nrightarrow 0$ as $n \to \infty$, for any $u \in \mathscr{C}^r[-1,1]$.

Let $\mathscr{K}$ be the integral operator defined by

$$(\mathscr{K}u)(s) = \int_{-1}^{1} \kappa(s,t)\, u(t)\, dt, \quad s \in [-1,1]. \tag{8}$$

Thus, equation (1) can be writing in operator form as

$$(I - \mathscr{K})u = f. \tag{9}$$

In the Kantorovich method [3,12], the integral equation (9) is approximated by

$$(I - \pi_n \mathscr{K})u_n = f, \tag{10}$$

while the iterated solution is defined by

$$\widetilde{u}_n = \mathscr{K}u_n + f \tag{11}$$

and it converges to u at a faster rate than the approximation u_n does.

Writing the Kantorovich-collocation solution u_n^C obtained by using π_n^C as

$$u_n^C = f + \sum_{i=0}^{n} a_i \ell_i, \quad s \in [-1,1],$$

equation (10) is equivalent to the following linear system of equations of size $n+1$

$$a_i - \sum_{j=0}^{n} a_j \int_{-1}^{1} \kappa(\tau_i,t)\, \ell_j(t)\, dt = \int_{-1}^{1} \kappa(\tau_i,t)\, f(t)\, dt, \quad i = 0,1,\dots,n. \tag{12}$$

Similarly, for the orthogonal projection, the Kantorovich–Galerkin solution of equation 10 is given by

$$u_n^G = f + \sum_{i=0}^{n} b_i L_i, \quad s \in [-1,1]$$

and the coefficients b_i are solutions of the linear system of equations

$$b_i - \sum_{j=0}^{n} b_j \langle \mathscr{K}L_j, L_i \rangle = \langle \mathscr{K}f, L_i \rangle, \quad i = 0,1,\dots,n. \tag{13}$$

In practice, the integrals in (2), (11), (12) and (13) are evaluated by an appropriate quadrature formula. This issue will be treated more extensively in Section 4.

3. Convergence rates

Let $p \geqslant 0$ and $\kappa \in \mathscr{C}^p[-1,1]^2$ then $R(\mathscr{K}) \subset \mathscr{C}^p[-1,1]$. Thus, if $f \in \mathscr{C}^p[-1,1]$ then the exact solution u of (1) belongs to $\mathscr{C}^p[-1,1]$. We set

$$D^{i,j}\kappa(s,t) = \frac{\partial^{i+j}\kappa}{\partial s^i \partial t^j}(s,t), \quad \|\kappa\|_{p,\infty} = \max\left\{\left\|D^{i,j}\kappa\right\|_\infty : i,j = 0,1,\dots,p\right\}$$

and

$$\|u\|_{p,\infty} = \max\left\{\|u^{(j)}\|_\infty \colon j = 0,1,\dots,p\right\}.$$

The next theorem provides the error estimate between u_n and the exact solution u.

Theorem 1. *Assume that* $\kappa \in \mathscr{C}^r[-1,1]^2$ *and that the inverse of* $(I - \mathscr{K})$ *exists and is uniformly bounded. Then for a sufficiently large* n*, the operators* $(I - \pi_n \mathscr{K})^{-1}$ *exist and*

$$\|(I - \pi_n \mathscr{K})^{-1}\| \leqslant A \tag{14}$$

for a suitable constant A *independent of* n*. Moreover,*

$$\begin{aligned} \|u - u_n\| &\leqslant \|(I - \pi_n \mathscr{K})^{-1}\| \|(I - \pi_n)\mathscr{K} u\|, \\ &\leqslant A \|(I - \pi_n)\mathscr{K} u\|. \end{aligned} \tag{15}$$

Proof. Using the fact that $\mathscr{K}$ is compact and $\|\mathscr{K} - \pi_n \mathscr{K}\| \to 0$ pointwise in $\mathscr{C}[-1,1]$, then from [1, Lemma 12.1.4] the operator $(I - \pi_n \mathscr{K})^{-1}$ exists and is uniformly bounded.

For the estimate (15), multiply $(I - \mathscr{K})u = f$ by π_n, and then rearrange to obtain

$$(I - \pi_n \mathscr{K})u = \pi_n f + (u - \pi_n u).$$

Subtract $(I - \pi_n \mathscr{K})u_n = f$ to get

$$\begin{aligned} (I - \pi_n \mathscr{K})(u - u_n) &= (I - \pi_n)(u - f), \\ &= (I - \pi_n)\mathscr{K} u. \end{aligned} \tag{16}$$

Taking norms the result follows. ∎

Throughout the paper, we assume that C is a generic constant which is independent of n. Now, we prove the theorem which establishes the rate of convergence of the approximation u_n to the exact solution u.

Theorem 2. *Assume that* $\kappa \in \mathscr{C}^r[-1,1]^2$ *and* $f \in \mathscr{C}[-1,1]$*. Let* u*,* u_n *be the solutions of* (10) *and* (11) *respectively. Then there exists a positive constant* C *independent of* n *such that*

$$\|u - u_n\|_\infty \leqslant C n^{\beta - r}. \tag{17}$$

Proof. From (7),

$$\begin{aligned} \|(I - \pi_n)\mathscr{K} u\|_\infty &\leqslant c_1 n^{\beta - r} \|(\mathscr{K} u)^{(r)}\|_\infty, \\ &\leqslant c_1 n^{\beta - r} \|\kappa\|_{r,\infty} \|u\|_\infty. \end{aligned}$$

Hence combining with (14) and (15) the estimate (17) is proved. ∎

The next theorem establishes the superconvergence of the iterated solution $\widetilde{u}_n$ to the exact solution u.

Theorem 3. *Let* $\widetilde{u}_n$ *be the iterated approximation of* u*. Assume that* $\kappa \in \mathscr{C}^r[-1,1]^2$*. Then for* n *large enough,*

$$\|u - \widetilde{u}_n\| \leqslant \|(I - \mathscr{K}\pi_n)^{-1}\| \|\mathscr{K}(I - \pi_n)\mathscr{K} u\|. \tag{18}$$

Proof. From the definition of the solutions u_n, $\widetilde{u}_n$, and estimate (16)

$$\begin{aligned} u - \widetilde{u}_n &= \mathscr{K}(u - u_n) \\ &= \mathscr{K}(I - \pi_n \mathscr{K})^{-1}(I - \pi_n)\mathscr{K} u. \end{aligned} \tag{19}$$

Moreover, it is shown in [1, Chap. 3] that

$$(I - \mathscr{K}\pi_n)^{-1} = [I + \mathscr{K}(I - \pi_n \mathscr{K})^{-1}\pi_n], \tag{20}$$

$$(I - \pi_n \mathscr{K})^{-1} = [I + \pi_n (I - \mathscr{K}\pi_n)^{-1}\mathscr{K}], \tag{21}$$

which implies that

$$(I - \pi_n \mathscr{K})^{-1}\pi_n = \pi_n (I - \mathscr{K}\pi_n)^{-1}$$

and therefore,

$$\mathscr{K}(I - \pi_n \mathscr{K})^{-1} = (I - \mathscr{K}\pi_n)^{-1}\mathscr{K}.$$

We now deduce the bound (18) from (19) and the above equality. ∎

For the rest of the paper, we set

$$z = \mathscr{K}u \quad \text{and} \quad \kappa_s(t) = \kappa(s,t), \quad s,t \in [-1,1].$$

Theorem 4. *Let $\pi_n^G \colon \mathscr{C}[-1,1] \to \mathbb{X}_n$ be the orthogonal projection and let $\widetilde{u}_n^G$ be the iterated Kantorovich–Galerkin solution defined by (11). Assume that $\kappa \in \mathscr{C}^r[-1,1]^2$ and $f \in \mathscr{C}[-1,1]$. Then there exists a positive constant C independent of n such that*

$$\|u - \widetilde{u}_n^G\|_\infty \leqslant Cn^{-2r}.$$

Proof. It follows from Theorem 3 that to estimate $\|u - \widetilde{u}_n^G\|_\infty$ it is necessary to estimate $\|\mathscr{K}(I - \pi_n^G)\mathscr{K}u\|$. Using the fact that π_n^G is the orthogonal projection from $\mathscr{C}[-1,1]$ to $\mathbb{X}_n$, we obtain for each $s \in [-1,1]$

$$\begin{aligned}\big[\mathscr{K}(I - \pi_n)\mathscr{K}u\big](s) &= \int_{-1}^{1} \kappa(s,t)\big(z - \pi_n^G z\big)(t)\,dt,\\ &= \langle \kappa_s - \pi_n^G \kappa_s, z - \pi_n^G z\rangle.\end{aligned}$$

Hence using the Cauchy–Schwarz inequality and (6),

$$\begin{aligned}\big\|\mathscr{K}(I - \pi_n^G)\mathscr{K}u\big\|_\infty &\leqslant \max_{s\in[-1,1]} \big\|\kappa_s - \pi_n^G\kappa_s\big\|_{L^2} \big\|z - \pi_n^G z\big\|_{L^2},\\ &\leqslant (c_1)^2 n^{-2r} \max_{s\in[-1,1]} \big\|\kappa_s^{(r)}\big\|_{L^2}\big\|z^{(r)}\big\|_{L^2},\\ &\leqslant 2(c_1)^2 \left(\|\kappa\|_{r,\infty}\right)^2 \|u\|_\infty n^{-2r}.\end{aligned}$$

This result, together with (18), completes the proof. ∎

Theorem 5. *Let $\pi_n^C \colon \mathscr{C}[-1,1] \to \mathbb{X}_n$ be the interpolatory projection defined by (3) and let $\widetilde{u}_n^C$ be the iterated Kantorovich-collocation solution defined by (11). Assume that $\kappa \in \mathscr{C}^r[-1,1]^2$ and $f \in \mathscr{C}[-1,1]$. Then there exists a positive constant C independent of n such that*

$$\|u - \widetilde{u}_n^C\|_\infty \leqslant Cn^{-r}. \tag{22}$$

Moreover, we have the following superconvergence estimate for u_n^C at the collocation points

$$\max_{0\leqslant i\leqslant n} \big|u(\tau_i) - u_n^C(\tau_i)\big| \leqslant Cn^{-r}.$$

Proof. Since

$$\left[\mathscr{K}\left(I - \pi_n^C\right)\mathscr{K}u\right](s) = \int_{-1}^{1} \kappa(s,t)\left(z - \pi_n^G z\right)(t)\,dt, \quad s \in [-1,1],$$

then, taking supremum and using the Cauchy–Schwarz inequality we get

$$\begin{aligned}\big\|\mathscr{K}\left(I - \pi_n^C\right)\mathscr{K}u\big\|_\infty &\leqslant \max_{s\in[-1,1]} \|\kappa_s\|_{L^2} \big\|z - \pi_n^C z\big\|_{L^2}\\ &\leqslant c_1\sqrt{2}\|\kappa\|_{r,\infty}\|z^{(r)}\|_{L^2} n^{-r}\\ &\leqslant 2c_1(\|\kappa\|_{r,\infty})^2\|u\|_\infty n^{-r}.\end{aligned}$$

Thus, by (18) the estimate (22) is proved.

Now, applying π_n^C to both sides of equations (10) and (11), we have that

$$\begin{aligned}\pi_n^C u_n &= \pi_n^C \mathscr{K}u_n^C + \pi_n^C f\\ &= \pi_n^C \widetilde{u}_n,\end{aligned}$$

and this yields

$$u_n^C(\tau_i) = \widetilde{u}_n^C(\tau_i), \quad i = 0,1,\ldots,n.$$

Hence, the required result follows from (22). ∎

Remark 1. According to (20) and (21), we can choose to show the existence of either $(I-\pi_n\mathscr{K})^{-1}$ or $(I-\mathscr{K}\pi_n)^{-1}$, whichever is the more convenient, and the existence of the other inverse will follow immediately. Bounds on one inverse in terms of the other can also be given by using (20) and (21).

Remark 2. If the right hand side of the operator equation is less smooth than the kernel of the integral operator, then it can be shown that the Kantorovich solution has a higher order of convergence than the Galerkin/collocation solution. Also we can prove that the iterated Kantorovich method had a faster convergence than the iterated Galerkin/collocation methods.

4. Discrete methods

In practice, the integrals in the definitions of $\mathscr{K}$ and π_n^G involved in equations (2) and (8) are not computed exactly. It is necessary to evaluate them by a numerical quadrature formula giving rise to discrete and iterated discrete Legendre–Kantorovich methods, respectively. To introduce these discrete methods, we consider a quadrature formula defined by

$$\int_{-1}^{1} f(t)\,dt \simeq \sum_{i=1}^{M} \omega_i f(t_i), \tag{23}$$

where the weights are such that

$$\omega_i > 0, \quad i=1,2,\dots,M$$

and the number of nodes is written simply M, with the dependence on n understood implicitly. We suppose that this formula has precision $2n$, that is

$$\int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{M} \omega_i f(t_i),$$

for all polynomials of degree $\leqslant 2n$. As a consequence we have for any $f\in\mathscr{C}^r[-1,1]$, $n\geqslant r$ (see [13])

$$\left|\int_{-1}^{1} f(t)\,dt - \sum_{i=1}^{M} \omega_i f(t_i)\right| \leqslant c_2 n^{-r}\big\|f^{(r)}\big\|_\infty,$$

where c_2 is a constant independent of n. Following Golberg [13] and Sloan [14] the discrete inner product is defined as

$$\langle f,g\rangle_M = \sum_{i=1}^{M} \omega_i\, f(t_i)\, g(t_i), \quad f,g\in\mathscr{C}[-1,1]. \tag{24}$$

Let $\mathscr{Q}_n^G\colon \mathscr{C}[-1,1]\to\mathbb{X}_n$ be the hyperinterpolation operator defined by Sloan [14] as

$$\mathscr{Q}_n^G u = \sum_{i=0}^{n} \langle u,\varphi_i\rangle_M \varphi_i(s) \tag{25}$$

and satisfies

$$\langle \mathscr{Q}_n^G u,\varphi_i\rangle_M = \langle u,\varphi_i\rangle_M, \quad i=0,1,\dots,n.$$

The following crucial properties of $\mathscr{Q}_n^G$ are quoted from Sloan [14] (see also [9]).

Lemma 2. *Let $\mathscr{Q}_n^G$ be the hyperinterpolation operator defined by (25). Then*

$$\big\|\mathscr{Q}_n^G u\big\|_{L^2} \leqslant \sqrt{2}\|u\|_\infty,$$
$$\big\|u-\mathscr{Q}_n^G u\big\|_{L^2} \leqslant 2\sqrt{2}\inf_{\phi\in\mathbb{X}_n}\|u-\phi\|_\infty.$$

Moreover, for any $u\in\mathscr{C}^r[-1,1]$,

$$\big\|u-\mathscr{Q}_n^G u\big\|_{L^2} \leqslant c_1 n^{-r}\|u^{(r)}\|_\infty,$$

where c_1 is a constant independent of n and $n\geqslant r$.

Note that for any $u \in \mathscr{C}^r[-1,1]$, we have also (see [8])

$$\left\|u - \mathscr{Q}_n^G u\right\|_\infty \leqslant c_1 n^{-r+1} \|u^{(r)}\|_\infty, \ n \geqslant r, \tag{26}$$

where c_1 is a constant independent of u and n. Moreover, from [9] $\mathscr{Q}_n^G$ satisfies

$$\left\langle u - \mathscr{Q}_n^G u, u - \mathscr{Q}_n^G u\right\rangle_M^{\frac{1}{2}} \leqslant c_1 \sqrt{2} n^{-r} \|u^{(r)}\|_\infty, \tag{27}$$

where c_1 is a constant independent of n and $n \geqslant r$.

For our convenience, from now onwards we write $\mathscr{Q}_n^G$ or $\mathscr{Q}_n^C = \pi_n^C$ as $\mathscr{Q}_n$. Using the numerical integration method (23), the Nyström approximation of the integral operator $\mathscr{K}$ is defined as

$$(\mathscr{K}_n u)(s) = \sum_{i=1}^{M} \omega_i \, \kappa(s, t_i) \, u(t_i).$$

The error bound (23) implies that for $\kappa \in \mathscr{C}^r[-1,1]$,

$$\|(\mathscr{K} - \mathscr{K}_n)u\|_\infty \leqslant c_2 n^{-r} \|\kappa\|_{r,\infty} \|u^{(r)}\|_\infty, \tag{28}$$

where c_2 is a constant independent of n. With the above notations, a discrete version of the approximate equation (10) is defined by

$$(I - \mathscr{Q}_n \mathscr{K}_n) z_n = f, \tag{29}$$

while the iterated discrete solution is given by

$$\widetilde{z}_n = \mathscr{K}_n z_n + f. \tag{30}$$

Thus, the integral operator $\mathscr{K}$ is replaced by $\mathscr{K}_n$ and the orthogonal projection π_n^G is replaced by the hyperinterpolation operator $\mathscr{Q}_n^G$. Notice that in the case $M = n+1$, and the quadrature points used in discrete inner product (24) and the collocation nodes in (3) are the same i.e. $\tau_{i-1} = t_i$, the operator $\mathscr{Q}_n^G$ coincides with the interpolatory projection operator $\mathscr{Q}_n^C$. Hence, to achieve the desired rates of convergence of the regular solution $\widetilde{u}_n^G$, we assume from now onwards that $M > n+1$.

Now we establish the convergence rates for the approximate solutions.

Theorem 6. *Assume that $\kappa \in \mathscr{C}^r[-1,1]^2$ and $f \in \mathscr{C}[-1,1]$. Let u, z_n be the solutions of (9) and (29) respectively. Then there exists a positive constant C independent of n such that*

$$\|u - z_n\|_\infty \leqslant C n^{\gamma - r}, \tag{31}$$

where $\gamma = 1$ for the hyperinterpolation operator and $\gamma = \frac{1}{2}$ for the interpolatory projection.

Proof. We write

$$(I - \mathscr{Q}_n \mathscr{K}_n)u = \mathscr{Q}_n f + (u - \mathscr{Q}_n u) + \mathscr{Q}_n(\mathscr{K} - \mathscr{K}_n)u.$$

Subtracting $(I - \mathscr{Q}_n \mathscr{K}_n) z_n = f$, one can obtain

$$\begin{aligned}(u - z_n) &= (I - \mathscr{Q}_n \mathscr{K}_n)^{-1}[(I - \mathscr{Q}_n)\mathscr{K} u + \mathscr{Q}_n(\mathscr{K} - \mathscr{K}_n)u], \\ &= (I - \mathscr{Q}_n \mathscr{K}_n)^{-1}[(I - \mathscr{Q}_n)\mathscr{K}_n u + (\mathscr{K} - \mathscr{K}_n)u].\end{aligned} \tag{32}$$

Taking norms, it follows from (15) that

$$\|u - z_n\|_\infty \leqslant 2A \max\left\{\|(I - \mathscr{Q}_n)\mathscr{K}_n u\|_\infty, \|(\mathscr{K} - \mathscr{K}_n)u\|_\infty\right\}.$$

On other hand, since the quadrature formula has precision $2n > n \geqslant r \geqslant 1$, choosing $f(x) = 1$ in (23)

$$2=\int_{-1}^{1} ds=\sum_{i=1}^{M}\omega_i$$

and using the fact that $\omega_j>0$, we obtain

$$\begin{aligned}\|(\mathscr{K}_n u)^{(r)}\|_\infty &= \sup_{s\in[-1,1]}\left|(\mathscr{K}_n u)^{(r)}(s)\right| = \sup_{s\in[-1,1]}\left|\sum_{i=1}^{M}\omega_i\frac{\partial^r\kappa}{\partial s^r}(s,t_i)u(t_i)\right|\\ &\leqslant 2\|\kappa\|_{r,\infty}\|u\|_\infty. \end{aligned}\tag{33}$$

Hence by (7) and (26), one can get

$$\begin{aligned}\|(I-\mathscr{Q}_n)\mathscr{K}_n u\|_\infty &\leqslant c_1 n^{\gamma-r}\|(\mathscr{K}_n u)^{(r)}\|_\infty,\\ &\leqslant 2c_1 n^{\gamma-r}\|\kappa\|_{r,\infty}\|u\|_\infty.\end{aligned}\tag{34}$$

Now combining (28), (32) and (34), we obtain the error bound (31). ∎

Theorem 7. *Let $\mathscr{Q}_n^G\colon \mathscr{C}[-1,1]\to\mathbb{X}_n$ be the hyperinterpolation operator and let $\widetilde{z}_n^G$ be the iterated discrete Kantorovich–Galerkin solution defined by (30). Assume that $\kappa\in\mathscr{C}^r[-1,1]^2$ and $f\in\mathscr{C}[-1,1]$. Then there exists a positive constant C independent of n such that*

$$\|u-\widetilde{z}_n^G\|_\infty\leqslant Cn^{-2r}.$$

Proof. Using (30) and the fact that

$$u=(\mathscr{K}-\mathscr{K}_n)u+\mathscr{K}_n u+f$$

we get

$$u-\widetilde{z}_n=(\mathscr{K}-\mathscr{K}_n)u+\mathscr{K}_n(u-z_n).$$

Moreover, by (32)

$$\begin{aligned}u-\widetilde{z}_n &= (\mathscr{K}-\mathscr{K}_n)u+\mathscr{K}_n(I-\mathscr{Q}_n\mathscr{K}_n)^{-1}[(I-\mathscr{Q}_n)\mathscr{K}_n u+(\mathscr{K}-\mathscr{K}_n)u],\\ &= (\mathscr{K}-\mathscr{K}_n)u+(I-\mathscr{Q}_n\mathscr{K}_n)^{-1}\mathscr{K}_n[(I-\mathscr{Q}_n)\mathscr{K}_n u+(\mathscr{K}-\mathscr{K}_n)u].\end{aligned}$$

Taking bounds,

$$\|u-\widetilde{z}_n\|_\infty\leqslant(1+2A)\max\left\{\|\mathscr{K}_n(I-\mathscr{Q}_n)\mathscr{K}_n u\|_\infty,\|(\mathscr{K}-\mathscr{K}_n)u\|_\infty,\|\mathscr{K}_n(\mathscr{K}-\mathscr{K}_n)u\|_\infty\right\},\tag{35}$$

Using the orthogonality of $\mathscr{Q}_n^G$, Cauchy–Schwarz inequality and estimate (27), we obtain for $s\in[-1,1]$

$$\begin{aligned}|\mathscr{K}_n(I-\mathscr{Q}_n^G)u(s)| &= \left|\sum_{i=1}^{M}\omega_i\kappa(s,t_i)\big(u-\mathscr{Q}_n^G u\big)(t_i)\right|\\ &= \left\langle\kappa_s,u-\mathscr{Q}_n^G u\right\rangle_M\\ &= \left\langle\kappa_s-\mathscr{Q}_n^G\kappa_s,u-\mathscr{Q}_n^G u\right\rangle_M\\ &= \left|\sum_{i=1}^{M}\omega_i\big(\kappa_s-\mathscr{Q}_n^G\kappa_s\big)(t_i)\big(u-\mathscr{Q}_n u\big)(t_i)\right|\\ &\leqslant \left(\sum_{i=1}^{M}\omega_i\big[\big(\kappa_s-\mathscr{Q}_n^G\kappa_s\big)(t_i)\big]^2\right)^{\frac12}\left(\sum_{i=1}^{M}\omega_i\big[\big(u-\mathscr{Q}_n^G u\big)(t_i)\big]^2\right)^{\frac12}\\ &= \left\langle\kappa_s-\mathscr{Q}_n^G\kappa_s,\kappa_s-\mathscr{Q}_n^G\kappa_s\right\rangle_M^{\frac12}\left\langle u-\mathscr{Q}_n^G u,u-\mathscr{Q}_n^G u\right\rangle_M^{\frac12}\\ &\leqslant 2(c_1)^2 n^{-2r}\|u^{(r)}\|_\infty\|\kappa\|_{r,\infty}.\end{aligned}$$

Hence (33) implies that

$$\begin{aligned}\left\|\mathscr{K}_n\left(I-\mathscr{Q}_n^G\right)\mathscr{K}_n u\right\|_\infty &\leqslant 2(c_1)^2 n^{-2r}\left\|\left(\mathscr{K}_n u\right)^{(r)}\right\|_\infty \|\kappa\|_{r,\infty}\\ &\leqslant 4(c_1)^2 n^{-2r}(\|\kappa\|_{r,\infty})^2\|u\|_\infty.\end{aligned} \tag{36}$$

Again by (28) and (33) we get

$$\begin{aligned}\|\mathscr{K}_n(\mathscr{K}-\mathscr{K}_n)u\|_\infty &\leqslant 2\|\kappa\|_{0,\infty}\|(\mathscr{K}-\mathscr{K}_n)u\|_\infty\\ &\leqslant 2c_2 n^{-r}\|\kappa\|_{0,\infty}\|\kappa\|_{r,\infty}\|u^{(r)}\|_\infty.\end{aligned} \tag{37}$$

The result now follows immediately by combining (28), (35), (36) and (37). ■

Theorem 8. *Let $\mathscr{Q}_n^C\colon \mathscr{C}[-1,1]\to\mathbb{X}_n$ be the interpolatory projection and let $\widetilde{z}_n^G$ be the iterated discrete Kantorovich-collocation solution defined by* (30)*. Assume that $\kappa\in\mathscr{C}^r[-1,1]^2$ and $f\in\mathscr{C}[-1,1]$. Then there exists a positive constant C independent of n such that*

$$\|u-\widetilde{z}_n^C\|_\infty \leqslant Cn^{-r}.$$

Proof. To prove the above bound we need to estimate the first term in the right hand side of (35). Then rewrite it as

$$\begin{aligned}\mathscr{K}_n(I-\mathscr{Q}_n^C)\mathscr{K}_n u &= (\mathscr{K}_n-\mathscr{K})(I-\mathscr{Q}_n^C)\mathscr{K}_n u+\mathscr{K}(I-\mathscr{Q}_n^C)\mathscr{K}_n u\\ &= (\mathscr{K}_n-\mathscr{K})\mathscr{K}_n u+(\mathscr{K}_n-\mathscr{K})\mathscr{Q}_n^C\mathscr{K}_n u+\mathscr{K}(I-\mathscr{Q}_n^C)\mathscr{K}_n u\end{aligned}$$

yields

$$\|\mathscr{K}_n(I-\mathscr{Q}_n^C)\mathscr{K}_n u\|_\infty \leqslant \|(\mathscr{K}-\mathscr{K}_n)\mathscr{K}_n u\|+\|(\mathscr{K}-\mathscr{K}_n)\mathscr{Q}_n^C\mathscr{K}_n u\|_\infty+\|\mathscr{K}(I-\mathscr{Q}_n^C)\mathscr{K}_n u\|_\infty$$

According to [5], for any $x_n\in\mathbb{X}_n$ and $n\geqslant r$

$$\|(\mathscr{K}-\mathscr{K}_n)x_n\|_\infty \leqslant 2\sqrt{2}c_3 n^{-r}\|\kappa\|_{r,\infty}\|x_n\|_{L^2}, \tag{38}$$

where c_3 is a constant independent of n. Using (4),

$$\begin{aligned}\|\mathscr{Q}_n^C\mathscr{K}_n u\|_{L^2} &\leqslant p\|\mathscr{K}_n u\|_{L^2}\\ &\leqslant p\sqrt{2}\|\mathscr{K}_n u\|_\infty\\ &\leqslant 2p\sqrt{2}\|\kappa\|_{0,\infty}\|u\|_\infty\end{aligned}$$

Thus, replacing x_n by $\mathscr{Q}_n^C\mathscr{K}_n u$ in (38),

$$\|(\mathscr{K}_n-\mathscr{K})\mathscr{Q}_n^C\mathscr{K}_n u\|_\infty \leqslant 8pc_3 n^{-r}\|\kappa\|_{r,\infty}\|\kappa\|_{0,\infty}\|u\|_\infty.$$

As in the proof of Theorem 4, we have by (33),

$$\begin{aligned}\|\mathscr{K}(I-\mathscr{Q}_n)\mathscr{K}_n u\|_\infty &\leqslant \max_{s\in[-1,1]}\|\kappa_s\|_{L^2}\|(I-\mathscr{Q}_n)\mathscr{K}_n u\|_{L^2}\\ &\leqslant c_1\sqrt{2}\|\kappa\|_{r,\infty}\|(\mathscr{K}_n u)^{(r)}\|_{L^2} n^{-r}\\ &\leqslant 2c_1\sqrt{2}(\|\kappa\|_{r,\infty})^2\|u\|_\infty n^{-r}.\end{aligned}$$

and therefore

$$\|\mathscr{K}_n(I-\mathscr{Q}_n^C)\mathscr{K}_n u\|_\infty \leqslant Cn^{-r}.$$

Now, the above estimate together with (28), (37) and (35) gives the desired result. ■

5. Numerical results

In this section, numerical examples are given to illustrate the results obtained in the previous sections. Note that, all required integrals were calculated by high precision with a 6-points Gauss quadrature rule. Let $\mathbb{X}_n$ denote the space of polynomials of degree $\leqslant n$. The computations are done for $n = 2, 3, 4, 5, 6$. We give the errors obtained by the discrete version of the Kantorovich method and its iterated version. In the case of interpolatory projection we give also the error at the collocation points

$$\max_{0\leqslant i\leqslant n} \left|u(\tau_i) - z_n^C(\tau_i)\right| = \max_i \left|u_i - z_{n,i}^C\right|.$$

Example 1. We choose the following Fredholm integral equation

$$u(s) - \int_{-1}^{1} e^{s+t} u(t)\, dt = f(s), \quad s \in [-1, 1],$$

where $f(s)$ is selected so that $u(s) = s$. The results are given in Tables 1 and 2.

Table 1. Kantorovich–Galerkin method.

n	$\lVert u - z_n^G \rVert_\infty$	$\lVert u - \widetilde{z}_n^G \rVert_\infty$
2	5.899404×10^{-2}	1.097404×10^{-3}
3	8.203238×10^{-3}	1.697011×10^{-5}
4	8.880388×10^{-4}	1.685388×10^{-7}
5	7.917584×10^{-5}	1.164501×10^{-9}
6	6.00258×10^{-6}	5.918313×10^{-12}

Table 2. Kantorovich–collocation method.

n	$\lVert u - z_n^C \rVert_\infty$	$\max_i \lvert u_i - z_{n,i}^C \rvert$	$\lVert u - \widetilde{z}_n^C \rVert_\infty$
2	6.379192×10^{-2}	1840809×10^{-3}	2.306218×10^{-3}
3	8.876835×10^{-3}	3.440448×10^{-5}	3.952963×10^{-5}
4	9.528347×10^{-4}	3.886120×10^{-7}	4.268367×10^{-7}
5	8426608×10^{-5}	2.957577×10^{-9}	3.164201×10^{-9}
6	6.346108×10^{-6}	1.624191×10^{-11}	1.708989×10^{-11}

Example 2. Consider

$$u(s) - \int_{-1}^{1} \sinh\left(\sqrt{2}s - 1\right) \cosh(t-1)\, u(t)\, dt = f(s), \quad s \in [-1, 1], \tag{39}$$

where $f \in \mathscr{C}[-1, 1]$ is so chosen that $u(s) = \sqrt{s+1}$ is a solution of (39). The results are given in Tables 3 and 4.

Table 3. Kantorovich–Galerkin method.

n	$\lVert u - z_n^G \rVert_\infty$	$\lVert u - \widetilde{z}_n^G \rVert_\infty$
2	8.451887×10^{-1}	1.354237×10^{-2}
3	1.317015×10^{-1}	2.925557×10^{-4}
4	2.433414×10^{-2}	4.076094×10^{-6}
5	2.430026×10^{-3}	3.959971×10^{-8}
6	3.069451×10^{-4}	2.834041×10^{-10}

Table 4. Kantorovich–collocation method.

n	$\|u - z_n^C\|_\infty$	$\max_i \|u_i - z_{n,i}^C\|$	$\|u - \tilde{z}_n^C\|_\infty$
2	1.204256×10^{-0}	2.718011×10^{-2}	3.765520×10^{-2}
3	1.983232×10^{-1}	7.878248×10^{-4}	9.625109×10^{-4}
4	3.474807×10^{-2}	1.352189×10^{-5}	1.547839×10^{-5}
5	3.593606×10^{-3}	1.547379×10^{-7}	1.705339×10^{-7}
6	4.560129×10^{-4}	1.269929×10^{-9}	1.366404×10^{-9}

6. Conclusion

The above tables illustrate that a high precision is reached even when the polynomials are of low degree and even when the right hand side f is only continuous. Note that the size of the linear systems is only $n+1$ and to obtain an accuracy of comparable order by piecewise polynomials a very much larger linear system are needed to be solved. It should be mentioned that the analysis given in this paper can be extended to the case of Green's kernels or weakly singular kernels.

[1] Atkinson K. E. The Numerical Solution of Integral Equations of the Second Kind. Cambridge University Press, Cambridge (1997).

[2] Atkinson K. E., Han W. Theoretical numerical analysis. Springer Verlag, Berlin (2005).

[3] Kantorovich L., Krylov V. Approximate Methods of Higher Analysis. Noordhoff, Groningen, The Netherlands (1964).

[4] Golberg M. Discrete Polynomial-Based Galerkin Methods for Fredholm Integral Equations. Journal of Integral Equations and Applications. **6** (2), 197–211 (1994).

[5] Kulkarni R. P., Nelakanti G. Iterated discrete polynomially based Galerkin methods. Applied Mathematics and Computation. **146** (1), 153–165 (2003).

[6] Nelakanti G., Panigrahi B. L. Legendre Galerkin method for weakly singular Fredholm integral equations and the corresponding eigenvalue problem. Journal of Applied Mathematics and Computing. **43**, 175–197 (2013).

[7] Long G., Nelakanti G., Sahani M. M. Polynomially based multi-projection methods for Fredholm integral equations of the second kind. Applied Mathematics and Computation. **215** (1), 147–155 (2009).

[8] Das P., Nelakanti G. Convergence analysis of discrete Legendre spectral projection methods for Hammerstein integral equations of mixed type. Applied Mathematics and Computation. **265**, 574–601 (2015).

[9] Das P., Nelakanti G. Error analysis of discrete legendre multi-projection methods for nonlinear Fredholm integral equations. Numerical Functional Analysis and Optimization. **38** (5), 549–574 (2017).

[10] Das P., Long G., Nelakanti G. Discrete Legendre spectral projection methods for Fredholm–Hammerstein integral equations. Journal of Computational and Applied Mathematics. **278**, 293–305 (2015).

[11] Chen C., Golberg M. Discrete projection methods for integral equations. Computational Mechanics Publications (1997).

[12] Sloan I. H. Four variants of the Gaterkin method for Integral equations of the second kind. IMA Journal of Numerical Analysis. **4** (1), 9–17 (1984).

[13] Golberg M. Improved convergence rates for some discrete Galerkin methods. Journal of Integral Equations and Applications. **8** (3), 307–335 (1996).

[14] Sloan I. H. Polynomial interpolation and hyperinterpolation over general regions. Journal of Approximation Theory. **83** (2), 238–254 (1995).

Метод Лежандра–Канторовича для інтегральних рівнянь Фредгольма другого роду

Аллуч К.[1], Аррaі М.[1], Боуда Х.[1], Тахрічі М.[2]

[1] *Університет Мохаммеда I, Team MSC, FPN, Лабораторія LAMAO, Надор, Марокко*
[2] *Університет Мохаммеда I, Team ANAA, EST, Лабораторія LANO, Уджда, Марокко*

У цій роботі розглядається поліноміальний метод Канторовича для чисельного розв'язування інтегрального рівняння Фредгольма другого роду з гладким ядром. Використовувана проекція є або ортогональною проекцією, або інтерполяційною проекцією з використанням базису поліномів Лежандра. Встановлено порядок збіжності запропонованого методу та порядок суперзбіжності ітераційних версій. Показано, що ці порядки збіжності справедливі у відповідних дискретних методах, які отримані заміною інтеграла квадратурою. Для ілюстрації теоретичних оцінок наведено числові приклади.

Ключові слова: *інтегральне рівняння Фредгольма, оператор проектування, поліном Лежандра, суперзбіжність, квадратурне правило, дискретний метод.*