

Enlarging the radius of convergence for Newton-like method in which the derivative is re-evaluated after certain steps

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Numerous attempts have been made to enlarge the radius of convergence for Newton-like method under the same set of conditions. It turns out that not only the radius of convergence but the error bounds on the distances involved and the uniqueness of the solution ball can more accurately be defined.

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1. Introduction

Let C be an open convex subset of Hilbert space B . The problem of computing solution s^* as of equation

$$F(x) = 0, \quad (1)$$

where $F: C \rightarrow B$ is differentiable in the sense of Fréchet is of extreme importance, since many applications reduce to solving (1). But closed form expression for s^* can be obtained only in special cases. That is why the solutions methods for equations (1) are iterative (mostly).

Recently a ball convergence result was given by Mărușter in [1, 2] for Newton-like method defined by iteration function H

$$\begin{aligned} y_{m+1} &= y_m - F'(x_n)^{-1} F(y_m), \quad m = 1, 2, \dots, k-1, \quad y_1 = x_n, \\ x_{n+1} &= H(x_n) = x_n - F'(x_n)^{-1} \sum_{m=1}^k F(y_m), \quad n = 0, 1, \dots \end{aligned} \quad (2)$$

This iteration can be considered as Picard-like one. Several well studied methods are special cases of (2).

Potra and Pták [3] (PP) studied (2) when $k = 2$. Moreover, in the scalar case Potra and Pták method was studied by Traub [4]. Ortega and Rheinboldt showed on infinite Euclidean space [5] that PP is of convergence order three. Notice also that PP is a special case of a multi-point method of the same order given in [6, 7] by Ezquerro and Hernández (EH). Moreover, Hernández and Romero [8] provided the radius of convergence for (EH) method. Furthermore, Cătișă [9] gave a radius of convergence for the general Picard iteration. Two-step methods for solving nonlinear problems were studied in [10, 11, 14].

Finally, Mărușter [2] gave a ball convergence result for method (2). Motivated by Mărușter's paper we provided a ball convergence under the same conditions with benefits:

- 1) at least as large radius of convergence (so at least as many initial points become available),
- 2) tighter error or bounds on $\|x_n - s^*\|$ become available (so at least a few iterations are needed to obtain certain error tolerance),
- 3) at least as precise information on the where about of the solution s^* is given.

This technique was applied to other iterative methods [12, 13, 15].

The ball convergence is given in Section 2, whereas the numerical experiments in Section 3.

2. Ball convergence

The aforementioned benefits are based on certain types of Lipschitz conditions. From now on we assume that s^* is a simple solution of equation (1) and $F: C \rightarrow B$ is Fréchet differentiable.

Definition 1. We say that operator F' satisfies the center-Lipschitz condition if there exists $l_0 > 0$ such that

$$\|F'(y) - F'(s^*)\| \leq l_0 \|y - s^*\| \quad (3)$$

for all $y \in C$.

Set $C_0 = C \cap U(s^*, \frac{1}{b_0 l_0})$, where b_0 will be determined later.

Definition 2. We say that operator F' satisfies the restricted-Lipschitz condition if there exists $l > 0$ such that

$$\|F'(y) - F'(x)\| \leq l \|y - x\| \quad (4)$$

for all $x, y \in C_0$.

Definition 3. We say that operator F' satisfies the Lipschitz condition if there exists $l_1 > 0$ such that

$$\|F'(y) - F'(x)\| \leq l_1 \|y - x\| \quad (5)$$

for all $x, y \in C$.

Remark 1. It follows from these definitions

$$C_0 \subset C \quad (6)$$

that

$$l_0 \leq l_1 \quad (7)$$

and

$$l \leq l_1. \quad (8)$$

We shall assume from now on that

$$l_0 \leq l. \quad (9)$$

Otherwise l_0 can replace l in all results that follow.

An upper bound on $\|F'(x)^{-1}\|$ was determined in [2] using (5). Indeed, assume

$$\|F'(s^*)^{-1}\| \leq b_0. \quad (10)$$

Then, for $v \in U(s^*, \frac{1}{b_0 l_1})$ we get by (5) and (10) that

$$\|F'(s^*)^{-1}(F'(v) - F'(s^*))\| \leq b_0 l_1 \|v - s^*\| < b_0 l_1 \frac{1}{b_0 l_1} = 1$$

so $F'(v)^{-1} \in L(B, B)$ by a lemma attributed to Banach [5, 16] linear operators and

$$\|F'(v)^{-1}\| \leq \frac{b_0}{1 - b_0 l_1 \|v - s^*\|}. \quad (11)$$

However, notice that the weaker (3) can be used to obtain instead of (5) the tighter estimate

$$\|F'(v)^{-1}\| \leq \frac{b_0}{1 - b_0 l_0 \|v - s^*\|}. \quad (12)$$

Let $\{r_n\}$, $n = 1, 2, \dots, k$ be a scalar sequence defined by

$$\begin{aligned} r_1 &= \bar{r}, \\ r_{n+1} &= \bar{a} r_n \left(1 + \frac{r_n}{2\bar{r}}\right). \end{aligned} \quad (13)$$

Then if $\bar{a} > \frac{2}{3}$ and $\bar{r} > 0$, sequence $\{r_n\}$ is strictly increasing [2]. The ball convergence is based on conditions (A).

Suppose:

(A₁) sequence $\{r_n\}$ is generated by (13) for $\bar{a} = a = \sqrt{3} - 1$, $\bar{r} = r \leq \frac{a}{b_0(l+al_0)}$ and $U(s^*, r_k) \subset C$;

(A₂) conditions (3), (4) and (10) hold.

Theorem 1. Suppose conditions (A) hold. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U(s^*, r_0)$, remains in $U(s^*, r_0)$ and converges to the unique solution $s^* \in U(s^*, r_0)$ of equation (1), where

$$r_0 = \frac{a}{b_0(l+al_0)}. \quad (14)$$

Moreover, the rate of convergence is at least $k+1$, and for all $x \in U(s^*, r_0)$

$$\|H(x) - s^*\| \leq \left(\frac{b_0(l+al_0)}{a} \right)^k \|x - s^*\|^{k+1}. \quad (15)$$

Proof. Simply use the proof of Corollary 3.2 in [2, page 19] with l, r_0 replacing l_1 ,

$$\bar{r}_0 = \frac{a}{(1+a)b_0l_1} \quad (16)$$

in [2], respectively.

We also use (12) instead of (11) and notice that

$$\frac{b_0}{l - b_0l_0r_0} = \frac{b_0(l+al_0)}{l}$$

by the definition of r_0 . ■

Remark 2. It follows from (7), (8), (14) and (16) that

$$\bar{r}_0 \leq r_0. \quad (17)$$

The corresponding to (15) given in [2] is

$$\|H(x) - s^*\| \leq \left(\frac{b_0(1+a)l_1}{a} \right)^k \|x - s^*\|^{k+1} \quad (18)$$

for all $x \in U(s^*, \bar{r}_0)$.

By completing (15) to $U(s^*, r_0)$ we see that new ratio of convergence is at least as small as the old one since

$$\frac{b_0(l+al_0)}{a} \leq \frac{b_0(1+a)l_1}{a}.$$

Clearly, Theorem 1 reduces to Corollary 3.2 in [2] if $l_0 = l = l_1$.

In view of (17) the uniqueness ball has been extended from $U(s^*, \bar{r}_0)$ to $U(s^*, r_0)$.

It turns out that we can do even better.

Proposition 3. Suppose that there exists $R \geq r_0$ such that

$$l_0R < 2. \quad (19)$$

Set $C_1 = C \cap U(s^*, R)$.

Then, the only solution of equation (1) in C_1 is s^* .

Proof. Let $z \in C_1$ with $F(z) = 0$. Define $T = \int_0^1 F'(s^* + \theta(z - s^*))d\theta$. Then, using (3) and (19), we get in turn that

$$\|F'(s^*)^{-1}(T - F'(s^*))\| \leq l_0 \int_0^1 \theta \|z - s^*\| d\theta \leq \frac{l_0R}{2} < 1,$$

so $z = s^*$ by $T^{-1} \in L(B, B)$ and the identity $0 = F(s^*) - F(z) = T(s^* - z)$. ■

Remark 3. In view of the above the benefits as stated in the introduction have been justified. Clearly, our technique extends the results of the aforementioned methods along the same lines. The efficiency of method (2) given in [2] is also improved, since the number of steps k after which F' is re-evaluated periodically increases under our approach. Notice also that

$$\frac{r_0}{\bar{r}_0} = \frac{1+a}{1+a\frac{l_0}{l}} \rightarrow 1+a = \sqrt{3}$$

as $\frac{l_0}{l} \rightarrow 0$. Hence, own technique increases the ball of convergence by almost $\sqrt{3}$ times.

3. Numerical examples

In this section we give some examples to confirm the theoretical results, namely that (17) is satisfied.

Example 1. Let $C = U(1, 1-p)$, $p \in (0, 0.9)$ and $B = \mathbb{R}$. Define function F on C by

$$F(x) = x^3 - p. \quad (20)$$

Since $s^* = \sqrt[3]{p}$ and $F'(x) = 3x^2$, we get that $b_0 = \frac{1}{3\sqrt[3]{p^2}}$, $l_1 = 6(2-p)$, $l_0 = 3(2-p + \sqrt[3]{p})$ and $l = 6 \min\left(2-p, s^* + \frac{1}{b_0 l_0}\right)$. Let $p = 0.725$. Then, we get $s^* \approx 0.8984$,

$$C = (0.7250, 1.2750), \quad b_0 \approx 0.4130, \quad l_1 = 7.6500, \quad l_0 \approx 6.5201,$$

$$C_0 \approx (0.7250, 1.2697), \quad l \approx 7.6181, \quad r_0 \approx 0.1430, \quad \bar{r}_0 \approx 0.1338.$$

So, all conditions in Remark 1 are satisfied.

Example 2. Let $C = U(0, 1)$ and $B = \mathbb{R}^3$. Define function F on C for $x = (\xi_1, \xi_2, \xi_3)^T$ by

$$F(x) = \left(e^{\xi_1} - 1, \frac{e-1}{2} \xi_2^2 + \xi_2, \xi_3 \right). \quad (21)$$

Since $s^* = (0, 0, 0)^T$ and $F'(x) = \text{diag}\{e^{\xi_1}, (e-1)\xi_2 + 1, 1\}$, we get that $b_0 = 1$, $l_1 = e$, $l_0 = e-1$ and $l = \max\left(e^{\frac{1}{e-1}}, e-1\right)$, $r_0 \approx 0.2402$, $\bar{r}_0 \approx 0.1555$. So, all conditions in Remark 1 are satisfied for this example too.

Notice that our radius r_0 is larger than $\bar{r}_0$ used in [1] as expected, since $l_0 < l_1$ and $l < l_1$. Hence, the benefits as claimed in the Introduction are also numerically justified.

4. Conclusion

The convergence analysis of Newton-like method is provided under classical center and restricted Lipschitz conditions. As a result, the convergence ball and the ball of uniqueness of the solution are enlarged, and the limits of error at the distances involved are determine more accurately. Moreover, the results were obtained under the same set of conditions as in the previous work. Numerical results that confirm the theoretical ones are given.

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Збільшення радіусу збіжності методу типу Ньютонa, в якому похідна обчислюється через декілька кроків

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Зроблено спробу збільшити радіус області збіжності методу типу Ньютонa за тих же умов, за яких метод вивчався раніше. Аналіз збіжності проведено за центральних та обмежених умов Ліпшиця. Крім радіусу області збіжності, вдалося отримати точніші оцінки похибки, а також більший радіус області єдиності розв'язку. Ці переваги є чисельно обґрунтованими.

Ключові слова: *радіус збіжності, метод типу Ньютонa, гільбертів простір.*