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WAVE PROCESSES IN THE CONSTRUCTIONS OF MECHANISMS AND BUILDINGS

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Abstract. In the article a methodology for researching wave processes in weakly nonlinear homogeneous one-dimensional systems – elements of machine and building structures is proposed. It is based on D'Alembert's method of constructing solutions of wave equations and the asymptotic method of nonlinear mechanics. Resonant and non-resonant cases are considered. In the non-resonant case, the influence of viscoelastic forces and small periodic disturbances on the oscillatory process is taken into account. In the resonant case, the influence of the disturbing force and the phase difference of natural oscillations is considered. Numerical methods for linear differential equations were used to analyze the obtained differential equations.

Keywords: wave process, elastic body, mathematical modeling, nonlinear homogeneous one-dimensional system

Introduction and Problem Statement

Various researchers have developed methods for constructing solutions to homogeneous boundary value problems for some classes of weakly nonlinear differential equations with partial derivatives that describe wave processes in one-dimensional systems (thread, string, rope, shaft, rod). It is based on:

a) the idea of presenting the solution of the corresponding undisturbing problem in the form of a wave, the frequency of which is related to the wave number, or the method of separating variables;

b) the principle of single-frequency oscillations of nonlinear systems with many degrees of freedom and distributed parameters [1];

c) the application of asymptotic methods of nonlinear mechanics [1, 2] for new classes of nonlinear systems.

A similar approach for describing wave processes in linear homogeneous as well as nonlinear elastic one-dimensional systems was considered, for example, in [1, 3–5]. In these works, the profile of the wave is described, as well as the influence of different natures of forces (including small nonlinear ones) on its main characteristics: amplitude, local frequency, and local wave number.

However, such important issues from the point of view of dynamics, as the influence of the contact of the medium with the source of disturbance of oscillations (boundary conditions), the initial state of the system (initial conditions of the corresponding differential equations) on the wave profile were not considered in works [3, 4].

Solving some of the issues of this complex problem about homogeneous boundary value problems for the nonlinear wave equation, which have an important applied value in the mathematical modeling of technological processes, for example, vibration volumetric processing, vibration transportation of products; dynamic processes in the elements of building structures and mechanisms are the subject of this work.

The purpose of the article is:

a) to use D'Alembert's method of constructing solutions of linear wave equations, to generalize the asymptotic Krylov–Bogolyubov–Mitropolsky method to a quasi-linear analog of the specified equation;

b) to show that in the case of wave processes close to the sinusoidal (cosinusoidal) form, the method based on the Fourier representation and the developed method are equivalent.

Main Material Presentation

A one-dimensional homogeneous elastic body, as an element of machine and structure structures, has a wide range of applications – in lifting and transport mechanisms, drilling rigs, power lines, etc. It is known [7] that the mathematical model of longitudinal oscillations of a one-dimensional homogeneous elastic body, the material of which satisfies a close to linear law of elasticity, is a differential equation:

$$u_{tt} - \alpha^2 u_{xx} = \varepsilon f\left(u, u_x, u_t, \chi\right), \tag{1}$$

where $u(x,t)$ – movement along the OX axis of the section with the coordinate x at any moment of time t; α – constant, which is expressed through the physical and mechanical parameters of the body; $\varepsilon f\left(u, u_x, u_t, \chi\right)$ – analytical, 2π – periodical by $\chi = \mu t$ function that characterizes the deviation of the elastic properties of the body material from the linear law, as well as the action of small compared to the restoring force of periodic external forces; μ – frequency of external periodic disturbance; ε – small parameter. This function can also take into account the influence of resistance forces and other dissipative forces on the dynamics of the process. Differential equation (1) also describes nonlinear torsional vibrations of shafts, transverse vibrations of a nonlinearly elastic string (rope). The methods of construction and research of its single- and multi-frequency solutions under homogeneous and disturbed boundary conditions are known and considered sufficiently for practical purposes, for example, in [1]. Below, we will consider a slightly different approach to the construction of asymptotic solutions of boundary value problems for the differential equation (1). It is based on D'Alembert's method of finding solutions of homogeneous wave equations and a physically based assumption: in the investigated homogeneous undisturbed medium, waves have a sinusoidal shape. This approach to the study of wave processes has received a new momentum of development in recent decades (see, for example, works [3–5]) and for many cases it is still one of the possible analytical methods for the study of dynamic processes in nonlinear systems.

Unperturbed equation. Such a case occurs when considering the above-mentioned problems in a linear formulation (without taking into account nonlinear forces) and it does not always adequately reflect the dynamics of the process. According to the general principles of constructing asymptotic solutions of nonlinear differential equations, solutions of the unperturbed ($\varepsilon = 0$) equation corresponding to (1), that is, equation:

$$u^0{}_{tt} - \alpha^2 u^0{}_{xx} = 0\,. \tag{2}$$

Depending on the method of fastening a one-dimensional homogeneous elastic body (for example, different types of fastening of power transmission lines on supports), equation (1), and therefore also (2), will be considered under boundary conditions:

$$u^0\left(x,t\right)\Big|_{x=0} = u^0\left(x,t\right)\Big|_{x=l} = 0\,, \tag{3}$$

$$u_x^0\left(x,t\right)\Big|_{x=0} = u_x^0\left(x,t\right)\Big|_{x=l} = 0\,, \tag{4}$$

$$u^0\left(x,t\right)\Big|_{x=0} = u_x^0\left(x,t\right)\Big|_{x=l} = 0\,, \tag{5}$$

which correspond to both fixed (boundary conditions (3)), both free (boundary conditions (4)), or a fixed beginning and a free (boundary conditions (5)) end of the environment. The solution of the formulated boundary value problems for equation (2) can be written in the form of superposition of direct and reflected waves:

$$u\left(x,t\right) = C_1 \cos\left(\kappa x + \omega t + \phi\right) + C_2 \cos\left(\kappa x - \omega t + \psi\right), \tag{6}$$

where ϕ, ψ, κ, ω, C_1, C_2 – constants, the content and appearance of which will be set below.

Note. In principle, from the mathematical point of view, it is not important to represent the wave in the form of a combination of sinusoidal or cosine components.

As follows from (2), relation (6) will transform the unperturbed equation (2) into the identity if the wavenumber and frequency are related by the dispersion relation:

$$\omega^2 - \alpha^2\kappa^2 = 0\,. \tag{7}$$

Satisfying one of the boundary conditions (3) – (5), in order to find the relationship between the initial phases of the direct and reflected waves and the wave number, i.e. the parameters ϕ, ψ, κ, we obtain a system of trigonometric equations:

$$\begin{gathered} C_1\cos(\omega t + \phi) + C_2\cos(-\omega t + \psi) = 0\,, \\ C_1\sin(\omega t + \phi) + C_2\sin(-\omega t + \psi) = 0 \end{gathered} \tag{8}$$

$$\begin{gathered} C_1\cos(\kappa l + \omega t + \phi) + C_2\cos(\kappa l - \omega t + \psi) = 0\,, \\ C_1\sin(\kappa l + \omega t + \phi) + C_2\sin(\kappa l - \omega t + \psi) = 0 \end{gathered} \tag{9}$$

$$\begin{gathered} C_1\cos(\omega t + \phi) + C_2\cos(-\omega t + \psi) = 0\,, \\ C_1\sin(\kappa l + \omega t + \phi) + C_2\sin(\kappa l - \omega t + \psi) = 0 \end{gathered} \tag{10}$$

Parameters C_1, C_2, ϕ, ψ are constant, therefore each of the obtained systems of equations must be fulfilled for an arbitrary value of t. The latter will take place if the phases of the direct and reflected waves are related by ratios:

$$\begin{gathered} C_1\cos\phi + C_2\cos\psi = 0\,,\ C_1\cos\phi - C_2\cos\psi = 0\,; \\ C_1\cos(\kappa l + \phi) + C_2\cos(\kappa l + \psi) = 0\,, C_1\sin(\kappa l + \phi) + C_2\sin(\kappa l + \psi) = 0\,; \end{gathered} \tag{11}$$

$$\begin{gathered} C_1\sin\phi - C_2\sin\psi = 0\,,\quad C_1\sin\phi + C_2\sin\psi = 0; \\ C_1\sin(\kappa l + \phi) - C_2\sin(\kappa l + \psi) = 0\,, C_1\cos(\kappa l + \phi) - C_2\cos(\kappa l + \psi) = 0\,; \end{gathered} \tag{12}$$

$$\begin{gathered} C_1\cos\phi + C_2\cos\psi = 0\,,\ C_1\sin\phi - C_2\sin\psi = 0\,; \\ C_1\cos(\kappa l + \phi) - C_2\cos(\kappa l + \psi) = 0\,, C_1\sin(\kappa l + \phi) + C_2\sin(\kappa l + \psi) = 0\,. \end{gathered} \tag{13}$$

As follows from (11)–(13), the obtained systems of equations have different from trivial ($C_1 = C_2 = 0$) solutions in the case when there are relationships of the form between the phases of the direct and reflected waves and the wave number:

$$\sin(\phi + \psi) = 0\,,\ sin(2\kappa l + \varphi + \psi) = 0\,. \tag{14}$$

From (7), (11)–(13) and (14), it follows: a) direct and reflected waves in the considered homogeneous medium move in opposite phases ($\phi + \psi = 0$) or the phases of the direct and reflected waves are shifted relative to each other by a constant amount ($\phi + \psi = \pi$); b) the wave numbers for the considered boundary conditions take values $\kappa = \frac{k\pi}{l}$, $\kappa = \frac{k\pi}{l}$, $\kappa = \frac{(2k+1)\pi}{2l}$, $k = 1, 2, \ldots$; c) the natural frequencies of wave packets in a medium with fixed and free ends are equal $\omega = \alpha\frac{k\pi}{l}$, and in the case of fixing only its beginning – $\omega = \alpha\frac{(2k+1)\pi}{l}$; d) the amplitudes of direct and reflected waves in a homogeneous medium are the same.

Thus, single-frequency wave processes in homogeneous one-dimensional structures, which are described by the differential equation (2) and satisfy the boundary conditions (3)–(5) can be written in the form:

$$u_k^0(x,t) = a_k\begin{cases} \cos\left(\frac{k\pi}{l}(x + \alpha t) + \phi_k\right) - \cos\left(\frac{k\pi}{l}(x - \alpha t) - \phi_k\right), \\ \cos\left(\frac{k\pi}{l}(x + \alpha t) + \phi_k\right) + \cos\left(\frac{k\pi}{l}(x - \alpha t) - \phi_k\right), \\ \cos\left(\frac{(2k+1)\pi}{2l}(x + \alpha t) + \phi_k\right) - \cos\left(\frac{(2k+1)\pi}{2l}(x - \alpha t) - \phi_k\right), \end{cases} \tag{15}$$

where a_k – amplitude parameter; ϕ_k – initial phase of direct and reflected waves.

It should be noted that the resulting expressions are easy to convert to the form:

$$\tilde{u}_k^0(x,t) = 2a_k \begin{cases} -\sin\frac{k\pi}{l}x\sin\left(\alpha\frac{k\pi}{l}t+\phi_k\right), \\ \cos\frac{k\pi}{l}x\cos\left(\alpha\frac{k\pi}{l}t+\phi_k\right), \\ \sin\frac{2(k+1)\pi}{2l}x\sin\left(\alpha\frac{2(k+1)\pi}{2l}t+\phi_k\right) \end{cases}, \tag{16}$$

and they, as expected, coincide with the known results [1,6], which can be found by the Fourier method for the corresponding boundary value problems for the unperturbed wave equation.

For the differential equation (2), additional initial conditions are considered (a mixed problem is posed):

$$u^0(x,t)|t=0=\varphi(x),\ \frac{\partial u^0(x,t)}{\partial t}|t=0=\vartheta(x), \tag{17}$$

where $\varphi(x)$ and $\vartheta(x)$ – known smooth functions that determine the initial state of the environment), then the solution of such a problem can be obtained in the form of a linear combination of its single-frequency solutions, i.e.

$$u^0(x,t)=\sum_n D_n u_n^0(x,t), \tag{18}$$

and the values of the integration constants are from the initial conditions (17). We will not consider the mixed problem for the perturbed equation (1) below, but will construct only the solution of the perturbed equation that satisfies one of the homogeneous boundary conditions in a form close to the k-th form of the unperturbed equation, and therefore, in the corresponding formulas, we will omit the index k for ease of writing.

The perturbed equation. This case takes place in the refined statement of the researched problem: in the case when nonlinear-elastic characteristics of the body are taken into account; viscoelastic forces that occur during oscillatory processes in solid bodies; small periodic disturbances. Developing the general idea of the asymptotic representation of the solution of quasi-linear differential equations, we will look for the solution of the perturbed equation (1) under homogeneous boundary conditions (3)–(5) in the form of an asymptotic series:

$$u(x,t)=\bar{u}^0(a,x,\theta)+\varepsilon u_1(a,x,\theta,\chi)+\varepsilon^2 u_2(a,x,\theta,\chi)+\ldots, \tag{19}$$

in which $\theta=\alpha\frac{k\pi}{l}t+\phi$ for boundary conditions (3) and (4) and $\theta=\alpha\frac{2(k+1)\pi}{2l}t+\phi$ for boundary conditions (5), $\bar{u}_0(a,x,\theta)=u_0^k(x,t)$ with the only difference that for the perturbed case the parameters a and ϕ are not constant but unknown slowly changing functions. To find the law of change of the latter for the perturbed equation (1), it is necessary to consider two cases: a) *non-resonant* – for which the frequency of natural oscillations of the undisturbed system ω_k is related to the frequency of the external disturbance by the aspect ratio $m\omega_k\neq n\mu$, $m,n=1,2,\ldots$ and resonant – for which there is a place $m\omega_k\approx n\mu$.

A non-resonant case. It is known [1] that in the non-resonant case the amplitude of oscillations of nonlinear systems does not depend on the frequency of the external periodic disturbance, therefore, as in [4], the laws of change of the slowly changing functions $a(x,t)$ and $\phi(x,t)$ will be specified in the form of differential equations:

$$\begin{aligned} a_t &= \varepsilon A_1(a)+\varepsilon^2 A_2(a)+\cdots, \\ a_x &= \varepsilon B_1(a)+\varepsilon^2 B_2(a)+\cdots, \\ \phi_t &= \varepsilon C_1(a)+\varepsilon^2 C_2(a)+\cdots, \\ \phi_x &= \varepsilon D_1(a)+\varepsilon^2 D_2(a)+\cdots, \end{aligned} \tag{20}$$

the right parts of which are still unknown functions and are found so that the asymptotic representation (19) satisfies the perturbed boundary value problem with the necessary accuracy.

As for the unknown functions $u_1(a,x,\theta,\chi)$, $u_2(a,x,\theta,\chi)$ – they must: a) be defined so that the asymptotic representation of the solution (19) satisfies the original equation (1) with the required accuracy; b) be 2π – periodic relative to θ,χ; c) satisfy the boundary conditions resulting from (3), (4) or (5). Below, without reducing the generality, we will consider only the boundary conditions (3) for the perturbed equation (1). For this purpose, by differentiating (19) with respect to time, we obtain:

$$u_t(x,t) = -a\frac{\alpha k\pi}{l}\left[\sin\left(\frac{k\pi}{l}x+\theta\right)-\sin\left(\frac{k\pi}{l}x-\theta\right)\right]+\varepsilon\left\{A_1(a)\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]\right.$$

$$\left.-C_1(a)\left[\sin\left(\frac{k\pi}{l}x+\theta\right)+\sin\left(\frac{k\pi}{l}x-\theta\right)\right]+\frac{\alpha k\pi}{l}\frac{\partial u_1}{\partial\theta}+\mu\frac{\partial u_1}{\partial\chi}\right\}+\varepsilon^2\ldots;$$

$$u_{tt}(x,t) = -a\left(\frac{\alpha k\pi}{l}\right)^2\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]+\varepsilon\left\{-2\frac{\alpha k\pi}{l}A_1(a)\times\right. \tag{21}$$

$$\times\left[\sin\left(\frac{k\pi}{l}x+\theta\right)+\sin\left(\frac{k\pi}{l}x-\theta\right)\right]-2a\frac{\alpha k\pi}{l}C_1(a)\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]+$$

$$\left.+\left(\frac{\alpha k\pi}{l}\right)^2\frac{\partial^2 u_1}{\partial\theta^2}+2\frac{\alpha k\pi}{l}\mu\frac{\partial^2 u_1}{\partial\theta\partial\chi}+\mu^2\frac{\partial^2 u_1}{\partial\chi^2}\right\}+\varepsilon^2\ldots$$

Similarly, by differentiating with respect to the second independent variable x, we have:

$$u_x(x,t) = -a\frac{k\pi}{l}\left[\sin\left(\frac{k\pi}{l}x+\theta\right)-\sin\left(\frac{k\pi}{l}x-\theta\right)\right]+\varepsilon\left\{B_1(a)\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]-\right.$$

$$\left.-D_1(a)\left[\sin\left(\frac{k\pi}{l}x+\theta\right)-\sin\left(\frac{k\pi}{l}x-\theta\right)\right]+\frac{\partial u_1}{\partial x}\right\}+\varepsilon^2\ldots;$$

$$u_{xx}(x,t) = -a\left(\frac{k\pi}{l}\right)^2\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]+\varepsilon\left\{-2\frac{k\pi}{l}B_1(a)\times\right. \tag{22}$$

$$\left.\left[\sin\left(\frac{k\pi}{l}x+\theta\right)+\sin\left(\frac{k\pi}{l}x-\theta\right)\right]2a\frac{k\pi}{l}D_1(a)\left[\cos\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right]+\frac{\partial^2 u_1}{\partial x^2}^2\right\}+\varepsilon^2.$$

The asymptotic representation of the solution (19), (20) will satisfy the original equation (1), if the coefficients at the same powers of the small parameter of its right and left parts, taking into account (21), (22), will be the same. The latter serves as a condition for determining unknown functions $u_1(a,x,\theta)$, $u_2(a,x,\theta)$, in particular, for finding $u_1(a,x,\theta)$, we obtain a linear differential equation:

$$\omega_k^2\frac{\partial^2 u_1}{\partial\theta^2}+2\omega_k\mu\frac{\partial^2 u_1}{\partial\chi\partial\theta}+\mu^2\frac{\partial^2 u_1}{\partial\chi^2}-\alpha^2\frac{\partial^2 u_1}{\partial x^2}=F_1(a,x,\theta)+4\omega_k\sin\frac{k\pi}{l}\times$$
$$\times\{[A_1(a)+\alpha^{-1}B_1(a)]\cos\theta+a[C_1(a)+\alpha^{-1}D_1(a)]\sin\theta\}, \tag{23}$$

where $F_1(a,x,\theta,\chi)=f[u,u_x,u_t,\chi]$, and $f[u,u_x,u_t,\chi]$ is function $f[u,u_x,u_t,\chi]$ value when:

$$u=a\left(os\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right),$$

$$u=a\left(os\left(\frac{k\pi}{l}x+\theta\right)-\cos\left(\frac{k\pi}{l}x-\theta\right)\right),$$

$$u_t=-\alpha\frac{k\pi}{l}\left(\sin\left(\frac{k\pi}{l}x+\theta\right)+\sin\left(\frac{k\pi}{l}x-\theta\right)\right).$$

Differential equations of a similar form are obtained for the second and subsequent approximations, only for them the functions $\boldsymbol{F_2(a,x,\psi,\chi)}, \boldsymbol{F_3(a,x,\psi,\chi)}, \ldots$ have a slightly more complicated form. Let's proceed to the solution of equation (23), that is, to finding the function $\boldsymbol{u_1(a,x,\theta,\chi)}$**.** The function $\boldsymbol{u(x,t)}$ will satisfy the boundary conditions (3) if $\boldsymbol{u_i(a,x,\theta,\chi)}$ presented in the form of series on the system of orthogonal functions $\{\boldsymbol{X_n(x)}\} = \left\{\mathbf{sin}\frac{\boldsymbol{n\pi}}{\boldsymbol{l}}\boldsymbol{x}\right\}$, in particular for the first approximation:

$$u_1(a,x,\theta,\chi) = \sum_{n=0} u_{1n}(a,\theta)\sin\frac{n\pi}{l}x. \tag{24}$$

Then the unknown 2π – periodic by θcoefficients $u_{1n}(a,\theta)$ are determined, as follows from (23), by a system of differential equations:

$$\omega_k^2\left(\frac{\partial^2 u_{1k}}{\partial\theta^2} + u_{1k}\right) + 2\omega_k\mu\frac{\partial^2 u_{1k}}{\partial\theta\partial\chi} = F_{1k}(a,\theta,\chi) + 4\omega_k \times$$
$$\times\{[A_1(a) + \alpha^{-1}B_1(a)]\cos\theta + a[C_1(a) + \alpha^{-1}D_1(a)]\sin\theta\},$$

when $n = k$

$$\omega_k^2\left(\frac{\partial^2 u_{1n}}{\partial\theta^2} + \frac{n^2}{k^2}u_{1n}\right) + 2\omega_k\mu\frac{\partial^2 u_{1k}}{\partial\theta\chi} = F_{1n}(a,\theta,\chi), \tag{25}$$

when $n \neq k$,

where $F_{1n}(a,\theta,\chi) = \frac{2}{l}\int_0^l F_1(a,x,\theta,\chi)\sin\frac{n\pi}{l}xdx.$

To determine the influence of nonlinear forces on the laws of change of the amplitude and phase of the wave process, that is, the functions $A_1(a), B_1(a), C_1(a), D_1(a)$, let's impose on the coefficient $u_{1k}(a,\theta,\chi)$, in addition to the condition of its 2π -periodicity by θ, an additional condition – the condition of the absence of proportional terms $\cos\theta$ and $\sin\theta$. This condition is equivalent to the condition of the absence of secular terms in the coefficients $u_{1k}(a,\theta,\chi)$. The latter allows obtaining a system of two algebraic equations with respect to four unknown functions $A_1(a), B_1(a), C_1(a), D_1(a)$:

$$A_1(a) + \alpha^{-1}B_1(a) = -\frac{\alpha}{4\pi^2\omega_k l}\times\int_0^l\int_0^{2\pi}\int_0^{2\pi} F_1(a,x,\theta,\chi)\sin\frac{k\pi}{l}x\cos d\,\chi\theta d\theta dx, ,$$
$$C_1(a) + \alpha^{-1}D_1(a) = -\frac{\alpha}{4\pi^2 a\omega_k l}\times\int_0^l\int_0^{2\pi}\int_0^{2\pi} F_1(a,x,\theta,\chi)\sin\frac{k\pi}{l}x\sin d\,\chi\theta d\theta dx. \tag{26}$$

Additional conditions connecting the above unknown functions can be obtained, for example, from the compatibility conditions for the functions $\varphi(x,t)$ (or $\psi(x,t)$) and $a(x,t)$, that is, from the conditions $\frac{\partial^2\phi}{\partial x\partial t} = \frac{\partial^2\phi}{\partial t\partial x}, \frac{\partial^2 a}{\partial x\partial t} = \frac{\partial^2 a}{\partial t\partial x}$. They, as follows from the equations of walking approximation for the specified functions, are transformed into the form:

$$A_1(a)\frac{dD_1(a)}{da} = B_1(a)\frac{dC_1(a)}{da},$$
$$A_1(a)\frac{dB_1(a)}{da} = B_1(a)\frac{dA_1(a)}{da}. \tag{27}$$

By direct integration of the second equation of the obtained ratios, we find: $A_1(a) = \gamma B_1(a)$ (γ – some constant), and from the first equation (27) taking into account the latter, it follows that $D_1(a) = \gamma C_1(a)$.

The coefficients themselves $u_{1n}(a,\psi)$, as follows from (23), are represented by Fourier series in the form:

$$u_{1k}(a,\theta,\chi) = \frac{1}{4\pi^2 l}\sum_{m,s}\frac{1}{(1-m^2)\omega_k^2 - 2ms\omega_k\mu}exp\,i\,(m\theta + s\chi)\times$$
$$\times\int_0^l\int_0^{2\pi}\int_0^{2\pi} F_1(a,x,\theta,\chi)\sin\frac{k\pi}{l}x\,exp - i(m\theta + s\chi)d\theta\,d\chi dx, \tag{28}$$

when $(1-m^2)\omega_k^2 - 2ms\omega_k\mu \neq 0.$

A resonant case. In contrast to the non-resonant case, in the resonant case the amplitude of the process significantly depends on the phase difference of the natural oscillations and the forcing force, i.e. on $\upsilon = \theta - \chi$. Below we consider only the case of the main resonance, i.e. $\omega_k = \mu$, for which the amplitude and phase difference are given by differential equations:

$$a_t = \varepsilon A_1(a,\upsilon) + \varepsilon^2 A_2(a,\upsilon) + \ldots,\ a_x = \varepsilon B_1(a,\upsilon) + \varepsilon^2 B_2(a,\upsilon) + \cdots;$$
$$\upsilon_t = \omega_k - \mu + \varepsilon C_1(a,\upsilon) + \varepsilon^2 C_2(a,\upsilon) + \ldots,\ \upsilon_x = \varepsilon D_1(a,\upsilon) + \varepsilon^2 D_2(a,\upsilon) + \ldots \tag{29}$$

Representing the solution of equation (1) in the resonant case in the form (19) with the only difference that the parameters a and $\theta = \upsilon + \chi$ are connected by the relations that follow from (29), we obtain:

$$u_t(x,t) = -a\frac{\alpha k\pi}{l}\left[\sin\left(\frac{k\pi}{l}x+\theta\right) - \sin\left(\frac{k\pi}{l}x-\theta\right)\right] +$$
$$\varepsilon\left\{A_1(a,\upsilon)\left[\cos\left(\frac{k\pi}{l}x+\theta\right) - \cos\left(\frac{k\pi}{l}x-\theta\right)\right] -\right.$$
$$\left.-C_1(a,\upsilon)\left[\sin\left(\frac{k\pi}{l}x+\theta\right) + \sin\left(\frac{k\pi}{l}x-\theta\right)\right] + \frac{\alpha k\pi}{l}\frac{\partial u_1}{\partial\theta} + \mu\frac{\partial u_1}{\partial\chi}\right\} + \varepsilon^2\ldots;$$

$$u_{tt}(x,t) = -a\left(\frac{\alpha k\pi}{l}\right)^2\left[\cos\left(\frac{k\pi}{l}x+\theta\right) - \cos\left(\frac{k\pi}{l}x-\theta\right)\right] + \varepsilon\left\{-\left[(\omega_k-\mu)\frac{\partial C_1(a,\upsilon)}{\partial\upsilon} + 2\frac{\alpha k\pi}{l}A_1(a,\upsilon)\right]\times\right.$$
$$\times\left[\sin\left(\frac{k\pi}{l}x+\theta\right) + \sin\left(\frac{k\pi}{l}x-\theta\right)\right] + \left[(\omega_k-\mu)\frac{\partial A_1(a,\upsilon)}{\partial\upsilon} - 2a\frac{\alpha k\pi}{l}C_1(a,\upsilon)\right]\times \tag{30}$$
$$\left.\times\left[\cos\left(\frac{k\pi}{l}x+\theta\right) - \cos\left(\frac{k\pi}{l}x-\theta\right)\right] + \left(\frac{\alpha k\pi}{l}\right)^2\frac{\partial^2 u_1}{\partial\theta^2} + 2\frac{\alpha k\pi}{l}\mu\frac{\partial^2 u_1}{\partial\theta\partial\chi} + \mu^2\frac{\partial^2 u_1}{\partial\chi^2}\right\} + \varepsilon^2\ldots.$$

$$u_{xx}(x,t) = -a\left(\frac{k\pi}{l}\right)^2\left[\cos\left(\frac{k\pi}{l}x+\theta\right) - \cos\left(\frac{k\pi}{l}x-\theta\right)\right]$$
$$+\ \varepsilon\left\{-2\frac{k\pi}{l}B_1(a,\upsilon)\left[\sin\left(\frac{k\pi}{l}x+\theta\right) + \sin\left(\frac{k\pi}{l}x-\theta\right)\right] -\right.$$
$$\left.-2a\frac{k\pi}{l}D_1(a,\upsilon)\left[\cos\left(\frac{k\pi}{l}x+\theta\right) - \cos\left(\frac{k\pi}{l}x-\theta\right)\right] + \frac{\partial^2 u_1}{\partial x^2}^2\right\} + \varepsilon^2\ldots$$

Thus, in the resonant case, the required functions are connected by a differential equation:

$$\omega_k^2\left(\frac{\partial^2 u_{1k}}{\partial\theta^2} + u_{1k}\right) + 2\omega_k\mu\frac{\partial^2 u_{1k}}{\partial\theta\partial\chi} =$$
$$= F_{1k}(a,\theta,\chi) + 4\omega_k\left\{\left[\frac{\omega_k-\mu}{2\omega_k}\frac{\partial C_1(a,\upsilon)}{\partial\upsilon} + A_1(a) + \alpha^{-1}B_1(a)\right]\cos\theta +\right.$$
$$\left.+a\left[C_1(a) + \alpha^{-1}D_1(a) - \frac{\omega_k-\mu}{2\omega_k a}\frac{\partial A_1(a,\upsilon)}{\partial\upsilon}\right]\sin\theta\right\}, \tag{31}$$

when $n = k$,

$$\omega_k^2\left(\frac{\partial^2 u_{1n}}{\partial\theta^2} + \frac{n^2}{k^2}u_{1n}\right) + 2\omega_k\mu\frac{\partial^2 u_{1k}}{\partial\theta\chi} = F_{1n}(a,\theta,\chi),$$

when $n \neq k$.

Proceeding in the same way as for the non-resonant case, with the only difference that in relations (31) θ is expressed through the phase difference υ and the phase of the external disturbance χ in the form of

$\theta = \upsilon + \chi$, we obtain a system of linear differential equations for determining the unknown functions $A_1(a,\upsilon), B_1(a,\upsilon), C_1(a,\upsilon), D_1(a,\upsilon)$:

$$\left[\frac{1-\mu}{2\omega_k}\frac{\partial C_1(a,\upsilon)}{\partial \upsilon} + A_1(a) + \alpha^{-1}B_1(a)\right]\cos\upsilon +$$
$$+a\left[C_1(a) + \alpha^{-1}D_1(a) - \frac{\omega_k - \mu}{2\omega_k a}\frac{\partial A_1(a,\upsilon)}{\partial \upsilon}\right]\sin\upsilon =$$
$$= -\frac{\alpha}{2\pi\omega_k l}\int_0^l\int_0^{2\pi} F_1(a,x,\chi+\upsilon,\chi)\sin\frac{k\pi}{l}x\cos\chi\, d\chi dx:$$
$$\left[\frac{\omega_k - \mu}{2\omega_k}\frac{\partial A_1(a,\upsilon)}{\partial \upsilon} - C_1(a) - \alpha^{-1}D_1(a)\right]\sin\upsilon - \tag{32}$$
$$-a\left[C_1(a) + \alpha^{-1}D_1(a) - \frac{\omega_k - \mu}{2\omega_k a}\frac{\partial A_1(a,\upsilon)}{\partial \upsilon}\right]\cos\upsilon =$$
$$= \frac{\alpha}{2\pi\omega_k l}\int_0^l\int_0^{2\pi} F_1(a,x,\chi+\upsilon,\chi)\sin\frac{k\pi}{l}x\sin\chi\, d\chi dx.$$

A system of differential equations (32) is obtained together with the compatibility conditions, which take the form:

$$A_1(a,\upsilon)\frac{\partial B_1(a,\upsilon)}{\partial a} + (\omega_k - \mu)\frac{\partial B_1(a,\upsilon)}{\partial \upsilon} = B_1(a,\upsilon)\frac{\partial A_1(a,\upsilon)}{\partial a} + (\omega_k - \mu)\frac{\partial A_1(a,\upsilon)}{\partial \upsilon},$$
$$A_1(a,\upsilon)\frac{\partial D_1(a,\upsilon)}{\partial a} + (\omega_k - \mu)\frac{\partial D_1(a,\upsilon)}{\partial \upsilon} = A_1(a,\upsilon)\frac{\partial C_1(a,\upsilon)}{\partial a} + (\omega_k - \mu)\frac{\partial C_1(a,\upsilon)}{\partial \upsilon}. \tag{33}$$

This determines the main characteristics of the wave in the medium. Note that the technique itself can be transferred to the case of boundary conditions (4) and (5). For the analysis of the resulting differential equations, in the general case, it is possible to use, for example, qualitative or numerical methods for linear differential equations. As for specific problems, for many cases it is significantly simplified, for example, in the case when the elastic forces satisfy the nonlinear technical law of elasticity [8], then $A_1(a) = 0$ and $B_1(a) = 0$; if only nonlinear resistance forces are taken into account, then $C_1(a) = 0$ and $D_1(a) = 0$.

As an example, consider the longitudinal oscillations of a hinged rod, the material of which satisfies the nonlinear technical law of elasticity [8] under the assumption that the resistance forces (internal friction forces) are small and proportional to the speed. The differential equation that describes the motion of such a one-dimensional system takes the form:

$$u_{tt} - \alpha^2 u_{xx} = \varepsilon(u_x)^2 u_{xx} - \beta u_t, \tag{34}$$

where $\alpha, \beta, \varepsilon$ – are expressed through the physical and mechanical characteristics of the rod. Boundary conditions for the differential equation (34), which correspond to the specified method of fixing, take the form (3). In the first approximation, in the regime of single-frequency oscillations close to the cosine form, the dynamic process is described in the rod by the dependence

$$u(x,t) = a\left(\cos\left(\frac{\pi}{l}(x+\alpha t) + \phi\right) - \cos\left(\frac{\pi}{l}(x-\alpha t) - \phi\right)\right) \tag{35}$$

in which the amplitude parameter and phase of oscillations are determined by differential equations:

$$a_t = -\beta\frac{a}{4};\ a_x = 0;\ \phi_t = \frac{3}{128}\frac{\varepsilon}{\alpha}a^2;\ \phi_x = 0. \tag{36}$$

Below, Fig. 1 shows the laws of change in frequency of oscillations over time.

The obtained dependencies show that in the first approximation of the solution of the given problem, the constant resistance force leads to the establishment of an isochronous dynamic process in the system.

As for the nonlinear restoring force, the latter can lead to either an increase in the frequency of natural oscillations (for rigid systems) or a decrease in it (for soft systems). In the case of neglecting the resistance force, the frequency of natural oscillations depends significantly on the amplitude of the initial disturbance.

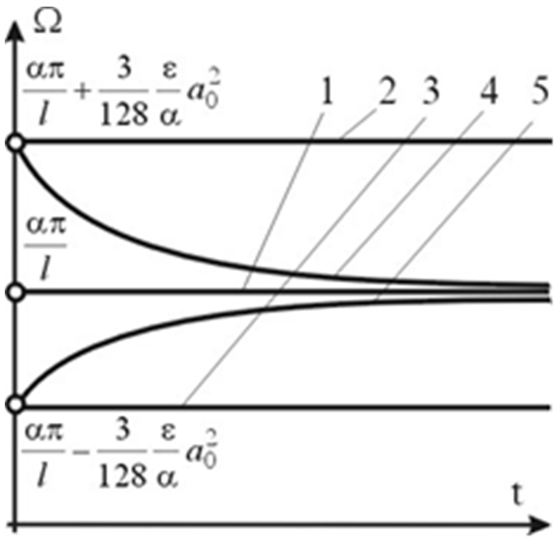

Fig. 1. Graphs of the dependence of the frequency of natural oscillations of the rod on time: 1 – linear case $(\boldsymbol{\beta = 0, \varepsilon = 0})$; 2, 3 – nonlinear case without taking into account the resistance force (2 – hard, 3 – soft nonlinear-elastic system); 4, 5 – non-linear case taking into account the resistance force (4 – for the case of a rigid system, 5 – for a soft system)

Conclusions

The proposed technique allows to investigate a wide range of important practical problems by mathematical modeling of the motion of one-dimensional bodies using boundary value problems for equation (1). Its main idea can be generalized to the case of more complex boundary conditions, as well as to some more complex mechanical systems, in particular to mechanical systems characterized by longitudinal motion, and such mechanical systems are much more difficult to analyze analytically.

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