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Chebyshev approximation of multivariable functions with the interpolation

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A method of constructing a Chebyshev approximation of multivariable functions by a generalized polynomial with the exact reproduction of its values at a given points is proposed. It is based on the sequential construction of mean-power approximations, taking into account the interpolation condition. The mean-power approximation is calculated using an iterative scheme based on the method of least squares with the variable weight function. An algorithm for calculating the Chebyshev approximation parameters with the interpolation condition for absolute and relative error is described. The presented results of solving test examples confirm the rapid convergence of the method when calculating the parameters of the Chebyshev approximation of tabular continuous functions of one, two and three variables with the reproduction of the values of the function at given points.

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1. Introduction

Let in some bounded area D of n-dimensional real space $\mathbb{R}^n$ the value of the function $f(X)$ are given on an arbitrary set of points $\Omega = \{X_j\}_{j=1}^{s}$. The function $f(X)$ for $s > m+1$ should be approximated by a generalized polynomial

$$F_m(a;X) = \sum_{i=0}^{m} a_i \varphi_i(X), \tag{1}$$

where $\varphi_i(X)$, $i = \overline{0,m}$ is a system of real functions linearly independent and continuous in D, a_i, $i = \overline{0,m}$ are unknown parameters: $\{a_i\}_{i=0}^{m} \in A$, $A \subseteq R^{m+1}$. The Chebyshev approximation of a function $f(X)$ by expression (1) with interpolation is that at the points of the subset $U = \{U_i\}_{i=1}^{k}$, $k < s$ $(U \subset \Omega)$, the expression $F_m(a;X)$ reproduces the value of the function $f(X)$

$$F_m(a;U_i) = f(U_i) = v_i, \quad i = \overline{1,k} \tag{2}$$

and the biggest approximation error

$$\Delta(a) = \max_{X\in\Omega} |f(X) - F_m(a;X)| \tag{3}$$

is the smallest possible out of the set of parameters A.

The class of expressions $F_m(a;x)$, satisfying condition (2) is characterized by the subset of parameters $\{a_i\}_{i=0}^{m} \in (A^* \subseteq A)$. If there is a point $a^* \in A^*$, at which the exact lower bound of the largest error of approximation is reached

$$\Delta(a^*) = \inf_{A^*} \Delta(a), \tag{4}$$

then the expression $F_m(a^*;x)$ is a Chebyshev approximation of a function $f(X)$ on a set of points Ω, that reproduces the given values of the function at a points $U - f(U_i) = v_i$, $i = \overline{1,k}$. An approximation that satisfies conditions (2) and (3) is called a Chebyshev approximation with the interpolation or with the interpolation condition [1–3].

The problem of finding the Chebyshev approximation with interpolation arises when the technical conditions require that the expression of the approximation at certain points reproduce the given values [4–6]. Such problems arise, in particular, during the design of measuring instruments [4, 7, 8], in which a certain value of the output signal of the sensor must correspond to a specific value of the measured value, description of transmission characteristics of automated control systems [9] and so on.

The Chebyshev approximation with the interpolation condition has certain features. In the case of approximation of a function of one variable, the property of alternate change of the error sign at the points of the Chebyshev alternation is not always fulfilled. This feature is noted in a number of works, in particular in [5, 10]. The characteristic property of the Chebyshev approximation with the condition was first established in [1], where it is substantiated that at the alternation points adjacent to the point of the condition, the sign of the deviation error must coincide. The properties of the Chebyshev approximation with the interpolation for the functions of one variable and algorithms for calculate its parameters are described in [1, 4, 10–13]. In [1, 4] the necessary and sufficient conditions for the existence of a Chebyshev approximation with the condition of expressions satisfying the Haar condition are established, and in [12–14] the characteristic properties are described and algorithms are proposed to determine the parameters of a Chebyshev approximation with the condition of some non-linear expressions. In [14], the alternative properties were established and methods for constructing the Chebyshev approximation of functions of one variable with interpolation of the values of the function and its derivatives were proposed. In the monograph [15] the problems of approximation of functions with the conditions in the form of inequalities are investigated. A modified algorithm for replacing alternative points in the construction of the Chebyshev approximation of one variable with interpolation is described in [1, 14].

We propose a method for constructing the Chebyshev approximation of multivariable functions by a generalized polynomial with interpolation as a boundary approximation in the norm of space L^p at $p \to \infty$. This method consists of sequentially constructing mean-power approximations with the reproduction of the values of the function at a given points. In the discrete case, we use the space E^p $(1 \leqslant p < \infty)$ with norm

$$\|\Delta\|_{E^p} = \Big(\sum_{X\in\Omega} |\Delta(X)|^p \Big)^{1/p}, \tag{5}$$

to estimate the error of the mean-power approximation. The limit value of the norm $\|\Delta\|_{E^p}$ at $p \to \infty$, similarly to the norm of the space L^p at $p \to \infty$, corresponds to the norm in the space of continuous functions $\|\Delta\|_C$ [3, 16].

2. The method for determining the Chebyshev approximation parameters of multivariable functions with interpolation

The construction of the Chebyshev approximation of the tabularly given of multivariable functions by a generalized polynomial with the interpolation is based on the idea of sequentially calculating the mean-power approximation in space E^p at $p = 2, 3, 4, \dots$ [17–19]. Suppose that for a function $f(X)$, given by the table on the set of points Ω, there is a Chebyshev approximation of the expression $F_m(a;X)$ with the interpolation at subset points U. The essence of the method of constructing the Chebyshev approximation of multivariable functions by expression $F_m(a;X)$ with interpolation we will illustrate by the example of calculating the parameters of the Chebyshev approximation with interpolation at only one point U_1. To construct such a Chebyshev approximation, we use the method of consecutive mean-power approximations by the expression

$$\bar{F}_m(a;X) = \sum_{i=1}^{m} a_i\varphi_i(X) + \left(v_1 - \sum_{i=1}^{m} a_i\varphi_i(U_1) \right) \frac{\varphi_0(X)}{\varphi_0(U_1)}. \tag{6}$$

The expression $\bar{F}_m(a;X)$ is derived from expression (1) with respect to interpolation condition (2) for only one point U_1, that is $f(U_1) = v_1$. This removes the parameter a_0 from expression (1). It is assumed that $\varphi_0(U)$ is not equal to zero. In practical implementation of this method, similarly to the modified Gaussian method for solving systems of linear equations with the choice of the principal element, in order to ensure the stability of the computational process, it is advisable to remove the parameter a_i $(i = \overline{0,m})$, where the value of the basis function $\varphi_i(U)$ at the point U_1 takes the largest absolute value.

The use of expression (6) to approximate reduces the problem of Chebyshev approximation with interpolation to the problem of constructing a Chebyshev approximation of multivariable functions $f(X)$ on a set of points $\Omega_u = \Omega \backslash U$ by a generalized polynomial of the form $\bar{F}_m(a;X)$ (6) with respect to unknown parameters a_i $(i = \overline{1,m})$. To calculate the values of these parameters, you can use an iterative scheme based on the method of least squares

$$\sum_{X\in\Omega_u} \rho_r(X)\left(f(X) - \bar{F}_m(a;X)\right)^2 \xrightarrow[a\in A]{} \min, \quad r = 1,\dots,p-2, \quad p = 2,3,4,\dots \tag{7}$$

with sequential refinement of the values of the weight function [18]

$$\rho_0(X) \equiv 1, \quad \rho_r(X) = \prod_{i=1}^{r} |\Delta_i(X)|, \quad r = 1,\dots p-2, \tag{8}$$

where $\Delta_k(X) = f(X) - \bar{F}_{m,k-1}(a;X)$, $k = \overline{1,r}$, $\bar{F}_{m,k-1}(a;X)$ is approximation of function $f(X)$ by expression (6) by least squares method with weight function $\rho_k(X)$ on the set of points $\Omega_u = \Omega \backslash U$.

The construction of the Chebyshev approximation of the function $f(X)$ consists in the sequential calculation its mean-power approximations by the expression $\bar{F}_m(a;X)$ for $p = 2,3,4,\dots$. The iterative process (7) with the weight function (8) provides sequential obtaining of mean-power approximations $\bar{F}_{m,r}(a;X)$, $r = 0,1,\dots$ of the function $f(X)$ in space E^{r+2} [18, 19], which in accordance with [16, 20] coincide to the Chebyshev approximation.

Completion of iterations (7) can be controlled by achieving some given accuracy ε

$$|\mu_{r-1} - \mu_r| \leqslant \varepsilon \mu_r, \tag{9}$$

where

$$\mu_r = \max_{X\in\Omega_u} \left| f(X) - \bar{F}_{m,r}(a;X) \right|. \tag{10}$$

If condition (9) is satisfied, the Chebyshev approximation of a continuous function given on a set of points with interpolation at a point by expression (1) takes the form

$$\bar{F}_m(a;X) = \bar{F}_{m,r}(a;X). \tag{11}$$

The obtained approximation (11) can be improved by an additive correction [16]

$$F_m(a;X) = \bar{F}_m(a;X) + \bar{a}_0. \tag{12}$$

The value of the correction $\bar{a}_0$ is defined as the solution of the one-parameter problem of the Chebyshev approximation of the function $f(X)$ by an expression $\bar{F}_m(a;X) + \bar{a}_0$ with absolute error at the set of points Ω

$$\bar{a}_0 = (\mu_{\max} + \mu_{\min})/2, \tag{13}$$

where

$$\mu_{\max} = \max_{X\in\Omega} \left(f(X) - \bar{F}_m(a;X)\right) \quad \text{and} \quad \mu_{\min} = \min_{X\in\Omega_u} \left(f(X) - \bar{F}_m(a;X)\right).$$

It should be noted that the adjusted approximation (12) of a continuous function $f(X)$ given on a set of points $X \in \Omega$ with interpolation at a point U_1 reduces the absolute error of the approximation by a value $\bar{a}_0$, but at the same time causes an error of the same value when reproducing a function $f(X)$ in the interpolation point U_1.

According to [18], the sequential refinement of the values of the weight function (8) taking into account the errors of reproduction of the values of the function $f(X)$ based on the results of all previous approximations least squares method (7), ensures the convergence of the iterative scheme (7), (8) when calculating mean-power approximations with interpolation. By setting the value ε in (9), it is possible to achieve the required accuracy of calculating the parameters of the Chebyshev approximation of the function $f(X)$ with interpolation.

The calculation of the parameters of the Chebyshev approximation by the generalized polynomial (1) with interpolation at several points is implemented according to a similar scheme. In this case, when forming a polynomial of the form (6) it is necessary to remove as many parameters from the polynomial (1) as there are interpolation points.

For example, the problem of constructing a Chebyshev approximation of multivariable functions $f(X)$ by a generalized polynomial (1) with interpolation at two points U_1 and U_2 is reduced to the problem of constructing a Chebyshev approximation of a function $f(X)$ at a set of points $\Omega_u = \Omega \backslash U$ by a generalized polynomial of the form

$$\bar{F}_m(a;X) = \sum_{i=2}^{m} a_i \varphi_i(X) + a_0 \varphi_0(X) + a_1 \varphi_1(X) \tag{14}$$

regarding unknown parameters a_i $(i = \overline{2,m})$. Assuming that the values $\varphi_0(U_1)$ and $\varphi_1(U_2)$ are not equal to zero, to determine the parameters a_0 and a_1 we obtain

$$a_0 = \left(v_1 - \sum_{i=1}^{m} a_i \varphi_i(U_1) \right) \Big/ \varphi_0(U_1), \tag{15}$$

$$a_1 = \frac{\varphi_0(U_1)\left(v_2 - \sum\limits_{i=2}^{m} a_i \varphi_i(U_2)\right) - \varphi_2(U_2)\left(v_1 - \sum\limits_{i=2}^{m} a_i \varphi_i(U_1)\right)}{\varphi_0(U_1)\varphi_1(U_2) - \varphi_1(U_1)\varphi_0(U_2)}, \tag{16}$$
$$v_1 = f(U_1), \quad v_2 = f(U_2).$$

3. Determining the parameters of the Chebyshev approximation of multivariable functions with the interpolation with relative error

If the continuous function $f(X)$ at the set of points Ω does not acquire values equal to zero, then a similar method can be used to calculate the Chebyshev approximation of $f(X)$ with interpolation at the point U_1 by expression (1) with a relative error. To construct a Chebyshev approximation of a function $f(X)$ with relative error and interpolation at a point U_1 we use an iterative scheme based on the least squares method (7) with a weight function

$$\rho_0(X) = \frac{1}{f^2(X)}, \quad \rho_r(X) = \prod_{i=1}^{r} |\Theta_i(X)|, \quad r = 1, \dots p-2, \quad p = 3, 4, \dots, \tag{17}$$

where

$$\Theta_k(X) = \frac{f(X) - \bar{F}_{m,k-1}(a;X)}{f(X)}, \quad k = \overline{1,r}, \tag{18}$$

and $\bar{F}_{m,k}(a;X)$ is the approximation of the function $f(X)$ by expression (6) by the method of least squares with the weight function $\rho_k(X)$.

In constructing the approximation with the relative error of completing the iterations (7) with the weighting function (17) it is possible to control the achievement of some necessary accuracy ε according to condition (12), in which

$$\mu_r(X) = \max_{X \in \Omega_u} |\Theta_{r+1}(X)|, \tag{19}$$

where $\Theta_{r+1}(X)$ is the error of the function reproduction $f(X)$ by the expression $\bar{F}_{m,r}(a;X)$, obtained by the method of least squares (7) at the iteration r.

As a result, the desired approximation of the continuous function $f(X)$ given on the set of points $X \in \Omega$ by expression (6) with relative error and interpolation at the point U_1 coincides with $\bar{F}_{m,r}(a;X)$

$$\bar{F}_m(a;X) = \bar{F}_{m,r}(a;X). \tag{20}$$

For approximation (20) it is possible to apply the corrective amendment b

$$F_m(a;X) = b\bar{F}_m(a;X), \tag{21}$$

where

$$b = \frac{2f(X_{\max})f(X_{\min})}{\bar{F}_m(a;X_{\min})f(X_{\max}) + \bar{F}_m(a;X_{\max})f(X_{\min})},$$

$X_{\min}$ and $X_{\max}$ are the points at which the relative error of the approximation $\Theta_{r+1}(X)$ (18) reaches the smallest and the largest value respectively. The value of the corrective amendment b is defined as the solution of the one-parameter problem of Chebyshev approximation of the function $f(X)$ by the expression $b\bar{F}_m(a;X)$ at the set of points $X \in \Omega_u$ with relative error

$$\max_{X\in\Omega_u} \left| \frac{f(X) - b\bar{F}_m(a;X)}{f(X)} \right| \underset{b}{\to} \min. \tag{22}$$

The results of the calculation of the Chebyshev approximation parameters with the interpolation for the test examples confirm the rapid convergence of the iterative process (7) with the weighting functions (8) and (15) when approximating the functions of one, two and three variables.

Example 1. Let us calculate the parameters of the Chebyshev approximation of the function $y(x) = \sqrt{0.1 + 2x + 3x^3}$ given at points x_i, $i = \overline{0,20}$, where $x_i = 0.1i$, by a polynomial of the second degree with interpolation at a point $U_1 = x_2 = 0.2$.

Using the proposed method at $\varepsilon = 0.003$ for 11 iterations (7) a polynomial

$$P_2(a;x) = 0.4557847182x^2 + 1.488032218x + 0.407479194, \tag{23}$$

was obtained for the function $y(x)$. Taking into account the corrective amendment $\bar{a}_0 = 0.0005614165$ it provides an absolute error of approximation 0.094260809.

The Chebyshev approximation of function $y(x)$ by a second degree polynomial with a given value at a point $U_1 = 0.2$, obtained by the iterative scheme of Remez with refinement of the points of alternation according to the modified Vallee–Poussin algorithm [1,14]

$$\overline{P}_2(a;x) = 0.409117888258291 + 1.48206220221688x + 0.458702739393668x^2, \tag{24}$$

provides the absolute approximation error 0.09289. The excess of the approximation error by polynomial (23) in comparison with the error of the Chebyshev approximation obtained by the Remez scheme (24) is equal to 0.00137, which is 1.475% of the error of the Chebyshev approximation obtained by the Remez scheme.

Error curve of the approximation of the function $y(x)$ by polynomial (23) is given in Fig. 1.

This error curve corresponds to the characteristic property of the Chebyshev approximation with interpolation at one point [1, 14]: it has three points at which reaches the largest deviation modulus, the values of the moduli of these deviations coincide and the sign of the error changes (except for points adjacent to the point of condition point $U_1 = 0.2$):

$$(0, -0.091251428),\ (0.9, -0.094260809),\ (2, 0.094260809). \tag{25}$$

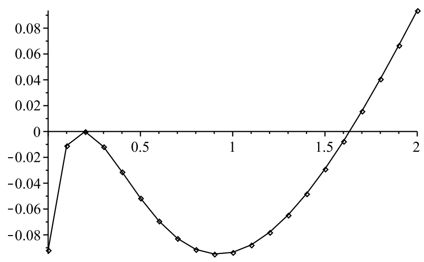

Fig. 1. Error curve of the approximation of the function $y(x)$ by polynomial (23) with interpolation at the point $U_1 = 0.2$.

In the extreme points adjacent to the interpolation point (in the first and second extreme points) the sign of the deviations coincides. The extreme points (25) coincide with the points of alternation obtained for the approximation of the function $y(x)$ according to the scheme of Remez (24). In the first extreme point (25), the value of the approximation error is slightly smaller in modulus. To achieve better alignment of the values of the modules of approximation errors at extreme points, you can increase the accuracy of the calculation of the Chebyshev approximation by reducing the value ε in (9).

The Chebyshev approximation of the function $y(x)$ with an absolute error of 0.09293 was obtained by method (7), (8) at $\varepsilon = 0.00004$ for 95 iterations

$$\widehat{P}_2\,(a;x) = 1.48264367x + 0.4584734726x^2 + 0.4088261344. \tag{26}$$

Value of the corrective amendment for approximation (24) is $\bar{a}_0 = -0.0001846355$. For the polynomial (26), the following error values were observed at the extreme points:

$$(0, -0.0925983684),\ (0.9, -0.092935947),\ (2, 0.092935948).$$

As the accuracy ε of calculating the approximation by a polynomial of the second degree increased, the extreme points did not change, and the values of the Chebyshev approximation error modules at these points almost leveled off: the deviation value at the first extreme point was less than the deviation at the other two points by 0.0003376, the discrepancy of values of deviations in extreme points makes 0.36%.

The Chebyshev approximation of the function $y(x)$ by a polynomial of second degree with interpolation at a point $U_1 = 0.2$ with relative error by method (7) with the weight function (17) for $\varepsilon = 0.003$ was obtained in 7 iterations. Polynomial

$$\overline{P}_2(a;x) = 0.184959529x^2 + 1.859979999x + 0.3440235337 \tag{27}$$

taking into account the corrective amendment $b = 0.9993638048$ provides a relative approximation error 9.378%.

The graph of the relative error of the approximation (27) also corresponds to the characteristic features of the Chebyshev approximation with interpolation at one point [1, 14]: in extreme points the relative error acquired the following values (in percent):

$$(0, -8.789793525),\ (0.6, -9.377984834),\ (2, 9.377984912).$$

The Chebyshev approximation of the function $y(x)$ by a polynomial of the second degree with relative error and interpolation at a point $U_1 = 0.2$, obtained by the iterative scheme of Remez with refinement of the points of alternation according to the modified Vallee–Poussin algorithm [1, 14]

$$\overline{P}_2(a;x) = 0.345662893101293 + 1.8533164802921x + 0.188806208325463x^2, \tag{28}$$

provides an approximation error 9.308%. In this case, the points of alternation of the approximation (28) coincide with the extreme points of approximation (27). The error of the polynomial approximation (27) exceeds the error of the Chebyshev approximation (28) obtained by the Remez scheme by only 0.07%

Example 2. Let us calculate the parameters of the Chebyshev approximation of the function $y_2(x) = \sqrt{0.1 + 2x + 3x^3 + 2.5x^4 + 1.75x^5}$, given at points x_i, $i = \overline{0, 40}$, where $x_i = 0.05i$, by a polynomial of the fourth degree with interpolation at points $U_1 = x_3 = 0.1$ and $U_2 = x_{37} = 1.85$.

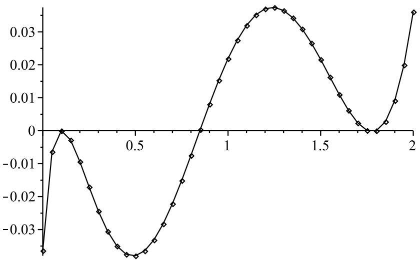

Fig. 2. Error curve of the approximation of the function $y_2(x)$ by polynomial (29) with interpolation at points $U_1 = 0.1$ and $U_2 = 1.85$.

Using the proposed method at $\varepsilon = 0.003$ for 11 iterations (7) a polynomial

$$P_4(a;x) = 0.3523318686 + 2.086566788x - 1.279891574x^2 + 2.291027573x^3 - 0.4144033331x^4, \tag{29}$$

was obtained for the function $y_2(x)$. Taking into account the corrective amendment $\bar{a}_0 = -0.0002580665$ it provides an absolute error of approximation 0.037684.

Error curve of the approximation of the function $y_2(x)$ by polynomial (29) is given in Fig. 2.

This error curve demonstrates the characteristic property of Chebyshev approximation with interpolation at two points [1, 14]: it has four extreme points at which reaches the largest deviation modulus, the values of the modules of these deviations coincide within a given accuracy and the deviation sign at these points alternates except for points adjacent to the interpolation points $U_1 = 0.1$ and $U_2 = 1.85$:

$$(0, -0.0361041026),\ (0.5, -0.037684047),\ (1.25, 0.037684047),\ (2, 0.03635155). \quad (30)$$

In the extreme points adjacent to the interpolation points are in the first and second extreme points, as well as in the third and fourth extreme points, the signs of deviations coincide. The extreme points (30) coincide with the points of alternation obtained to approximate the function $y_2(x)$ according to the Remez scheme with the replacement of the points of alternation according to the modified Vallee–Poussin algorithm [1, 14]. In the first extreme and fourth extreme points (30) the value of the approximation error is slightly smaller in modulus. To achieve a better alignment of the values of the modules of the approximation errors at extreme points, it is possible to increase the accuracy of the Chebyshev approximation by reducing the value ε in (9).

The Chebyshev approximation of the function $y_2(x)$ by a polynomial of the fourth degree with interpolation at points $U_1 = 0.1$ and $U_2 = 1.85$ was obtained by method (7)–(8) at $\varepsilon = 0.00004$ with an absolute error 0.037134448 with a corrective amendment $\bar{a}_0 = 0.000154495$ for 65 iterations

$$P_4(a; x) = 0.3523318686 + 2.086566788x - 1.279891574x^2 + 2.291027573x^3 - 0.4144033331x^4. \quad (31)$$

For the polynomial (31), the following error values were observed at the extreme points:

$$(0, -0.0369351072),\ (0.5, -0.037134448),\ (1.25, 0.037134448),\ (2, 0.03689639).$$

As the accuracy of calculating the approximation by a fourth-degree polynomial increased, the extreme points did not change, and the values of the Chebyshev approximation error modules at these points almost leveled off: the deviation at the fourth extreme point is less than the deviation at the second and third points by 0.0001993408 the discrepancy between the values of deviations at extreme points is 0.54%.

Example 3. Let us construct the Chebyshev approximation of the function $z(x, y) = \sqrt{1 + x^2 + y^2}$, given at points (x_i, y_j), $i = \overline{0, 10}$, $j = \overline{0, 10}$, where $x_i = 0.1i$, $y_j = 0.1j$, by a second-degree polynomial with respect to variables x and y with absolute error and interpolation at the point $(0.7, 0.7)$, that is, by reproducing the value — $z_1(0.7, 0.7) = \sqrt{1.98}$.

Since the function $z(x, y)$ and the interpolation point $(0.7, 0.7)$ are symmetric with respect to the arguments x and y, its approximation by a polynomial the second degree with respect to variables x and y must also be symmetric, that is for approximate the function $z(x, y)$ it is expedient to use a polynomial of the form

$$P_{2,2}(x, y) = a + b(x + y) + c\left(x^2 + y^2\right)$$

Using the proposed method at $\varepsilon = 0.003$ for 7 iterations (7), (8) a polynomial

$$P_{2,2}(a; x, y) = 0.9846166165 + 0.1155239943(x + y) + 0.2995600510\left(y^2 + x^2\right) - 0.06878210928xy, \quad (33)$$

was obtained for the function $z(x, y)$. Taking into account the corrective amendment $\bar{a}_0 = 0.0009089035$ it provides an absolute error of approximation of the function $z(x, y)$ is 0.0153833835.

Surface of the approximation error of the function $z(x, y)$ by polynomial (33) is given in Fig. 3.

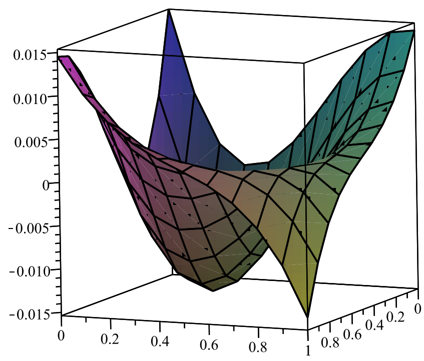

Fig. 3. Surface of approximation error of the function $z(x, y)$ by polynomial (33) with absolute error and interpolation at the point $(0.7, 0.7)$.

Chebyshev approximation of the function $z(x, y)$ by second degree polynomial in the form (32) with absolute error and interpolation at the point $(0.7, 0.7)$ at $\varepsilon = 0.00003$ is obtained in 75 iterations (7), (8)

$$P_{2,2}(a; x, y) = 0.984430447 + 0.1160263448(y + x) + 0.2992658491\left(y^2 + x^2\right) - 0.6740560715xy. \quad (34)$$

Polynomial approximation (34) reproduces the values of the function $z(x,y)$ with an absolute error 0.015569553 taking into account the corrective amendment $\bar{a}_0 = -0.00000561295$.

Let us find the Chebyshev approximation of the function $z(x,y)$ by a polynomial of the second degree of the form (32) with relative error and interpolation at a point $(0.7, 0.7)$. Using the proposed method at $\varepsilon = 0.003$ in 5 iterations (7), (17) for the function $z(x,y)$ a polynomial

$$\bar{P}_{2,2}(a;x,y) = 0.9873646925 + 0.1042095768(x+y) + 0.3066564207\left(y^2+x^2\right) - 0.05623979992xy \quad (35)$$

is obtained. Taking into account the corrective amendment $b = 0.999359803$ it provides the relative error of the approximation function $z(x,y)$ is 1.26%.

Example 4. Let us find the Chebyshev approximation of the function $z_3(x,y,t) = e^{-xyt}$ given at points (x_i, y_j, t_r), $i = \overline{0,10}$, $j = \overline{0,10}$, $r = \overline{0,10}$, where $x_i = 0.1i$, $y_j = 0.1j$, $t_r = 0.1r$ by a polynomial of the first degree for each of the variables x, y and t and a given value at a point $(0,0,0)$, that is, by reproducing the value $z_3(0,0,0) = 1$.

Using the proposed method (7), (8) at $\varepsilon = 0.003$ an approximating polynomial

$$\begin{aligned} P_1(a;x,y,t) = {} & 0.9983521186 - 0.03058442607x - 0.03058437606y - 0.03058428869t \\ & + 0.02446439166xy + 0.02446429362xt + 0.02446423364yt - 0.652122295xyt \end{aligned} \quad (36)$$

was obtained for 22 iterations. It provides an absolute error 0.04 of approximation of the function $z_3(x,y,t)$ with the adjusting amendment $\bar{a}_0 = -0.00164788145$.

The Chebyshev approximation of the function $z_3(x,y,t)$ a polynomial of first degree from variables x, y and t with relative error and given value at the point $(0,0,0)$ at $\varepsilon = 0.003$ obtained in 18 iterations:

$$\begin{aligned} \widehat{P}_1(a;x,y,t) = {} & 0.9985727597 - 0.05483632707x - 0.05483620694y - 0.05483570435t \\ & + 0.04896204689xt + 0.04896260434xy + 0.04896191445yt - 0.6358532879xyt. \end{aligned} \quad (37)$$

The relative error of this approximation with adjusting correction $b = 0.9985727594$ is equal to 6.214%.

4. Discussion

The calculation of the parameters of the Chebyshev approximation of multivariable functions with the condition is based on the construction of the boundary mean-power approximation. The least squares method with variable weight function was used for this purpose [18,19]. The possibility of such an approach is investigated in detail in [16]. In [17], a consistent refinement of the values of the weight function

$$\rho_0(X) \equiv 1, \quad \rho_r(X) = \prod_{i=1}^{r} |\Delta_i(X)|^2, \quad r = 1,2,\ldots \quad (38)$$

was proposed, where $\Delta_k(X) = f(X) - \bar{F}_{m,k-1}(a;X)$, $k = \overline{1,r}$, $\bar{F}_{m,k}(a;X)$ is approximation by least squares method of a function $f(X)$ with a weight function $\rho_k(X)$. Clarification of the values of the weight function by formula (38) in comparison with the α-method of E. Ya. Remez [16] takes into account the results of approximation in previous iterations.

As a result of further study of the convergence of mean-power approximation, it was proposed to refine the values of the weight function by formula (8) [18,21]. The application of the least squares method (7) with a variable weight function (8) provides consistent obtaining of mean-power approximations $\bar{F}_{m,r}(a;X)$, $r = 0,1,\ldots$, of the function $f(X)$. The value of the weight function at each iteration (7) increases in proportion to the modulus of approximation error

$$\mu_r(X) = \left|f(X) - \bar{F}_{m,r}(a;X)\right|, \quad (39)$$

which was obtained in the previous iteration.

The convergence of this method was substantiated in [18,21]. Since the point with the largest deviation value (39) corresponds to the proportionally largest value of the weight function (8), the application of such refinement of the weight function for iterations (7) causes a consistent decrease in the largest error of approximation of the function $f(X)$ on the set of points X ($X \in \Omega$). Therefore, the application of the iterative procedure (7), (8), leads to a consistent reduction of the error of reproduction of the function $f(X)$ by approximation $\bar{F}_{m,r}(a;X)$. This justifies the convergence of iterations.

In [22, 23] the convergence of the least squares method with the refinement of the weight function according to scheme (8) was confirmed when constructing the Chebyshev approximation of the functions of many variables by a rational expression.

Given the above, we note that the authors of [24] rightly hope to improve the convergence of AAA–Lawson algorithm using a consistent refinement of the values of the weight function by formula (8).

5. Conclusions

The parameters of the Chebyshev approximation of multivariable functions by a generalized polynomial with the interpolation with the smallest absolute error are calculated according to the iteration scheme (7)–(8), and with the smallest relative error according to the iteration scheme (7), (17). The method consists in sequential construction of mean-power approximations with an interpolation condition. The mean-power approximations are calculated using an iterative scheme based on the least-squares method (7) with the original weight function, the values of which are formed taking into account the results of approximation in previous iterations. It is possible to calculate the parameters of the Chebyshev approximation of multivariable functions with the condition with the required accuracy. Test examples confirm the sufficiently fast convergence of the proposed method when calculating the Chebyshev approximation parameters with both absolute and relative error.

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Чебишовське наближення функцій багатьох змінних з інтерполюванням

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Запропоновано метод побудови чебишовського наближення функції багатьох змінних узагальненим поліномом з відтворенням її значень у заданих точках. Він ґрунтується на послідовній побудові середньостепеневих наближень з врахуванням інтерполяційної умови. Середньостепеневе наближення обчислюється за ітераційною схемою на основі методу найменших квадратів зі змінною ваговою функцією. Описано алгоритм для обчислення параметрів чебишовського наближення з інтерполяційною умовою для абсолютної та відносної похибки. Подані результати розв'язування тестових прикладів підтверджують швидку збіжність методу під час обчислення параметрів чебишовського наближення таблично заданих неперервних функцій однієї, двох і трьох змінних з відтворенням значення функції у заданих точках.

Ключові слова: *чебишовське наближення з інтерполяційною умовою, функції багатьох змінних, середньостепеневе наближення, метод найменших квадратів, змінна вагова функція.*