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A mathematical study of the COVID-19 propagation through a stochastic epidemic model

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The COVID-19 is a major danger that threatens the whole world. In this context, mathematical modeling is a very powerful tool for knowing more about how such a disease is transmitted within a host population of humans. In this regard, we propose in the current study a stochastic epidemic model that describes the COVID-19 dynamics under the application of quarantine and coverage media strategies, and we give a rigorous mathematical analysis of this model to obtain an overview of COVID-19 dissemination behavior.

Keywords: *COVID-19; Brownian motion; stochastic epidemic model; coverage media; quarantine.*

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1. Introduction

The deterministic formulations analysis is very necessary and commonly used in the mathematical epidemiology, and it can be seen as the first tool for modeling new diseases spread and getting an overview of their asymptotic behavior. But, the real phenomena are not always deterministic and may be subject to some uncertainties and randomness due to fluctuations in the natural environment. Therefore, an adapted mathematical formulation that considers this stochasticity is required in the case of COVID-19. For this purpose, we will treat in this study a probabilistic version that incorporates proportional Gaussian white noises of the compartmental model presented in [1]. More precisely, we will analyze the following stochastic differential equations system:

$$\begin{cases} \mathrm{d}S = \left[\Lambda - \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S(I+\theta A) + \lambda Q - (\mu+q)S\right]\mathrm{d}t + \sigma_1 S \mathrm{d}B_1(t), \\ \mathrm{d}Q = \big[qS - (\mu+\lambda)Q\big]\mathrm{d}t + \sigma_2 Q\,\mathrm{d}B_2(t), \\ \mathrm{d}E = \left[\left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S(I+\theta A) - (\mu+\sigma)E\right]\mathrm{d}t + \sigma_3 E\,\mathrm{d}B_3(t), \\ \mathrm{d}A = \big[(1-p)\sigma E - (\mu+\varepsilon_A+\gamma_A+d_A)A\big]\mathrm{d}t + \sigma_4 A\,\mathrm{d}B_4(t), \\ \mathrm{d}I = [\sigma p E - (\mu+\varepsilon_I+\gamma_I+d_I)I]\,\mathrm{d}t + \sigma_5 I\,\mathrm{d}B_5(t), \\ \mathrm{d}H = \big[\varepsilon_I I + \varepsilon_A A - (\mu+d_H+\gamma_H)H\big]\mathrm{d}t + \sigma_6 H\,\mathrm{d}B_6(t), \\ \mathrm{d}R = \big[\gamma_H H + \gamma_I I + \gamma_A A - \mu R\big]\mathrm{d}t + \sigma_7 R\,\mathrm{d}B_7(t). \end{cases} \tag{1}$$

Here, $(\sigma_1, \dots, \sigma_7) \in \mathbb{R}_+^7$ designate the intensities of the mutually independent Brownian motions B_i $(i = 1, \dots, 7)$. These latter, and all the random variables that will be evoked in our analysis, are supposed to be defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ that is endowed with an usual filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ (it is increasing and right continuous while $\mathfrak{F}_0$ contains all $\mathbb{P}-$null sets). The analysis of the stochastic model (1) is divided into four parts. First, we check in detail its well-posedness in the sense that it admits one and only one solution which is global in time, positive, and bounded. Then, and under appropriate conditions, some interesting asymptotic properties are proved, namely: extinction and persistence in the mean. The theoretical results show that the dynamics of the perturbed COVID-19 model are determined by parameters that are closely related to the magnitude of the stochastic noise. Finally, we present some numerical illustrations to confirm our theoretical results and to show the impact of media intervention and quarantine strategies on the COVID-19 prevalence [2–5].

2. Well-posedness

In this subsection, we show that the model (1) is well posed, in the sense that if $S(0)$, $Q(0)$, $E(0)$, $A(0)$, $I(0)$, $H(0)$, and $R(0)$ are positive, then the system admits one and only one solution which is global in time, positive, and bounded.

Theorem 1. *For any initial value $X_0 \in \mathbb{R}^7_+$, there is a unique solution $X(t)$ to the system (1) on $t \geqslant 0$, and it will remain in $\mathbb{R}^7_+$ with probability one, which means that, if $\big(S(0), Q(0), E(0), A(0), I(0), H(0), R(0)\big)$ is in $\mathbb{R}^7_+$, then $\big(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\big) \in \mathbb{R}^7_+$ for all $t \geqslant 0$ almost surely (a.s. for short).*

Proof. In the system (1), the coefficients are continuously differentiable on their domains of definition, so they satisfy the local Lipschitz condition, and for this reason, there exists for any given initial value $X_0 \in \mathbb{R}^7_+$, a unique maximal local solution $X(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [6]. At this point, our goal will be to demonstrate that this solution is global, that is $\tau_e = \infty$ a.s.

To this purpose, let $k_0 \in \mathbb{N}$ be very large such that $X(0) \in [k_0^{-1}, k_0]$, and define for each integer $k \geqslant k_0$ the stopping time τ_k as follows:

$$\begin{aligned}\tau_k &= \inf\left\{t \in [0, \tau_e) \mid (\exists i \in \{1, \dots, 7\})\colon X_i(t) \notin \left(\tfrac{1}{k}, k\right)\right\} = \inf\left\{t \in [0, \tau_e) \mid X(t) \notin \left(\tfrac{1}{k}, k\right)^7\right\} \\ &= \inf\left\{t \in [0, \tau_e) \mid \min_{1\leqslant i\leqslant 7} X_i(t) \leqslant \tfrac{1}{k} \ \text{ or } \ \max_{1\leqslant i\leqslant 7} X_i(t) \geqslant k\right\}. \end{aligned} \tag{2}$$

Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, clearly, $(\tau_k)_{k\geqslant k_0}$ is increasing; hence, $\lim_{k\to\infty} \tau_k = \sup_{k\geqslant k_0} \tau_k$, and according to Lemma 2.11 of [7] $\sup_{k\geqslant k_0} \tau_k$ is a stopping time, then so is τ_∞. By adopting the convention $\inf \varnothing = \infty$ for the rest of this paper, we can easily affirm that $\tau_\infty \leqslant \tau_e$ a.s. Hence, $\tau_e = \infty$ a.s. will follow directly if we show that $\tau_\infty = \infty$ a.s., and that is exactly what we are going to do to finish the proof.

Assume that $\tau_\infty = \infty$ a.s. is untrue, then there exists a positive constant T such that $\mathbb{P}(\tau_\infty \leqslant T) > 0$.

Therefore, there exists an $\varepsilon > 0$ for which

$$\mathbb{P}(\tau_k \leqslant T) > \varepsilon \ \text{ for all } \ k \geqslant k_0. \tag{3}$$

Consider the $\mathcal{C}^2$function V defined for $x = (x_1, \dots, x_7) \in \mathbb{R}^7_+$ by

$$V(x) = \left[x_1 - a - a\ln\frac{x_1}{a}\right] + \sum_{i=2}^{7} \big(x_i - 1 - \ln(x_i)\big),$$

where a is a positive constant to be chosen suitably later. The nonnegativity of this function can be deduced from the following inequality: $x - 1 - \ln x \geqslant 0$, $\forall x > 0$.

Applying the multi-dimensional Itô's formula to $V(X(t))$, we obtain for all $k \geqslant k_0$ and $t \in [0, \tau_k)$

$$\begin{aligned}&\mathrm{d}V\big(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\big) \\ &\qquad = \mathcal{L}V\big(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\big)\,\mathrm{d}t + (S(t) - a)\sigma_1\,\mathrm{d}B_1(t) \\ &\qquad\quad + (Q(t) - 1)\sigma_2\,\mathrm{d}B_2(t) + (E(t) - 1)\sigma_3\,\mathrm{d}B_3(t) + (A(t) - 1)\sigma_4\,\mathrm{d}B_4(t) \\ &\qquad\quad + (I(t) - 1)\sigma_5\,\mathrm{d}B_5(t) + (H(t) - 1)\sigma_6\,\mathrm{d}B_6(t) + (R(t) - 1)\sigma_7\,\mathrm{d}B_7(t),\end{aligned}$$

where $\mathcal{L}V\colon \mathbb{R}^7_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}&\mathcal{L}V(S, Q, E, A, I, H, R) \\ &= \left(1 - \frac{a}{S}\right) \times \left[\Lambda - \left(\beta_1 - \beta_2\frac{I}{b + I}\right) S(I + \theta A) + \lambda Q - (\mu + q)S\right] \\ &\quad + \left(1 - \frac{1}{Q}\right) \times \big[qS - (\mu + \lambda)Q\big] + \left(1 - \frac{1}{E}\right) \times \left[\left(\beta_1 - \beta_2\frac{I}{b + I}\right) S(I + \theta A) - (\mu + \sigma)E\right]\end{aligned}$$

$$
\begin{aligned}
&+\left(1-\frac{1}{A}\right)\times\left[(1-p)\sigma E-(\mu+\varepsilon_A+\gamma_A+d_A)A\right]+\left(1-\frac{1}{I}\right)\times\left[\sigma pE-(\mu+\varepsilon_I+\gamma_I+d_I)I\right]\\
&+\left(1-\frac{1}{H}\right)\times\left[\varepsilon_I I+\varepsilon_A A-(\mu+d_H+\gamma_H)H\right]+\left(1-\frac{1}{R}\right)\times\left[\gamma_H H+\gamma_I I+\gamma_A A-\mu R\right]\\
&+\frac{1}{2}\left[a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2\right]\\
=\,&\Lambda-\mu(S+Q+E+A+I+H+R)-d_AA-d_II-d_HH\\
&+\left[-\frac{a\Lambda}{S}+a\beta_1(I+\theta A)-a\beta_2\frac{I(I+\theta A)}{b+I}-a\frac{\lambda Q}{S}+a(\mu+q)\right]+\left[-q\frac{S}{Q}+(\mu+\lambda)\right]\\
&+\left[-\left(\beta_1-\beta_2\frac{I}{b+I}\right)\frac{S}{E}(I+\theta A)+(\mu+\sigma)\right]+\left[-(1-p)\sigma\frac{E}{A}+(\mu+\varepsilon_A+\gamma_A+d_A)\right]\\
&+\left[-\sigma p\frac{E}{I}+(\mu+\varepsilon_I+\gamma_I+d_I)\right]+\left[-\varepsilon_I\frac{I}{H}-\varepsilon_A\frac{A}{H}+(\mu+\gamma_H+d_H)\right]\\
&+\left[-\gamma_H\frac{H}{R}-\gamma_I\frac{I}{R}-\gamma_A\frac{A}{R}+\mu\right]+\frac{1}{2}\left[a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2\right]\\
\leqslant\,&\Big[\Lambda+6\mu+\lambda+\sigma+\varepsilon_A+\gamma_A+d_A+\varepsilon_I+\gamma_I+d_I+d_H+\gamma_H+a(\mu+q)+\frac{1}{2}\big(a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2\\
&+\sigma_5^2+\sigma_6^2+\sigma_7^2\big)\Big]-\mu(S+Q+E+R)-\theta\beta_1\left(\frac{\mu+d_A}{\theta\beta_1}-a\right)A-\beta_1\left(\frac{\mu+d_I}{\beta_1}-a\right)I.
\end{aligned}
$$

By choosing $a=\frac{1}{2}\min\left\{\frac{\mu+d_A}{\theta\beta_1},\frac{\mu+d_I}{\beta_1}\right\}$, the coefficients of A and I will be negatives, therefore

$$
\begin{aligned}
\mathcal{L}V(S,Q,E,A,I,H,R)\leqslant\,&\Lambda+6\mu+\lambda+\sigma+\varepsilon_A+\gamma_A+d_A+\varepsilon_I+\gamma_I+d_I+d_H+\gamma_H+a(q+\mu)\\
&+\frac{1}{2}\left(a\sigma_1^2+\sum_{i=2}^{7}\sigma_i^2\right)\triangleq\mathcal{K}.
\end{aligned}
$$

Hence, we get for all $k\geqslant k_0$ and $t\in[0,\tau_k)$

$$
\begin{aligned}
&\mathrm{d}V\big(S(t),Q(t),E(t),A(t),I(t),H(t),R(t)\big)\\
&\quad\leqslant\mathcal{K}\,\mathrm{d}t+(S(t)-a)\sigma_1\,\mathrm{d}B_1(t)+(Q(t)-1)\sigma_2\,\mathrm{d}B_2(t)+(E(t)-1)\sigma_3\,\mathrm{d}B_3(t)\\
&\qquad+(A(t)-1)\sigma_4\,\mathrm{d}B_4(t)+(I(t)-1)\sigma_5\,\mathrm{d}B_5(t)+(H(t)-1)\sigma_6\,\mathrm{d}B_6(t)+(R(t)-1)\sigma_7\,\mathrm{d}B_7(t).
\end{aligned}
$$

Integrating from 0 to $\tau_k\wedge T$ and then taking the expectation on both sides of the above inequality leads to

$$\mathbb{E}\big[V(X(T\wedge\tau_k)\big]\leqslant V(X(0))+\mathcal{K}\mathbb{E}\left[\tau_k\wedge T\right]\leqslant V(X(0))+\mathcal{K}T. \tag{4}$$

We have $V(x)\geqslant 0$ for all $x>0$, then

$$\mathbb{E}\big[V(X(T\wedge\tau_k)\big]=\mathbb{E}\big[V(X(T\wedge\tau_k)\times\mathbb{1}_{\{\tau_k\leqslant T\}}\big]+\mathbb{E}\big[V(X(t\wedge\tau_k)\times\mathbb{1}_{\{\tau_k>T\}}\big]\geqslant\mathbb{E}\big[V(X(\tau_k)\times\mathbb{1}_{\{\tau_k\leqslant T\}}\big], \tag{5}$$

where $\mathbb{1}_A$ denotes the indicator function of a measurable set $A\in\mathfrak{F}$. Note that for every $\omega\in\{\omega\in\Omega\mid\tau_k(\omega)\leqslant T\}$, there is some component of $V\big(X(\tau_k)\big)$ equals to k or $\frac{1}{k}$ so

$$V\big(X(\tau_k)\big)\geqslant\left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{k}-a-a\ln\frac{1}{ka}\right)\wedge(k-1-\ln k)\wedge\left(\frac{1}{k}-1-\ln\frac{1}{k}\right).$$

Therefore

$$
\begin{aligned}
\mathbb{E}\big[V(X(\tau_k)\times\mathbb{1}_{\{\tau_k\leqslant T\}}\big]\geqslant\,&\mathbb{P}(\tau_k\leqslant T)\left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{k}-a-a\ln\frac{1}{ka}\right)\wedge(k-1-\ln k)\\
&\wedge\left(\frac{1}{k}-1-\ln\frac{1}{k}\right).
\end{aligned}\tag{6}
$$

Combining (4), (5) and (6) with (3), we conclude that

$$V(X(0))+\mathcal{K}T\geqslant\varepsilon\left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{k}-a-a\ln\frac{1}{ka}\right)\wedge(k-1-\ln k)\wedge\left(\frac{1}{k}-1-\ln\frac{1}{k}\right).$$

Letting $k\to\infty$ leads to the contradiction $V(X(0))+\mathcal{K}T=\infty$, which completes the proof. ∎

3. Extinction of COVID-19

In this section, we will give some conditions for the extinction of the model (1) expressed in terms of system parameters and intensities of noises. For the sake of simplicity, we will denote from now on the temporary mean $\frac{1}{t}\int_0^t \varphi(s)\,\mathrm{d}s$ of a continuous function φ by $\langle\varphi(t)\rangle$. Also, and for all $(x,y)\in\mathbb{R}^2$, we adopt the following notations $x\wedge y := \min(x,y)$ and $x\vee y := \max(x,y)$.

Definition 1 (Stochastic extinction). *For system* (1)*, the infected individuals* $E(t)$*,* $A(t)$ *and* $I(t)$ *are said to be stochastically extinct, or extinctive, if* $\lim_{t\to\infty} E(t)+I(t)+A(t)=0$ *almost surely.*

Before stating the result to be proved, we must firstly give the following useful lemma.

Lemma 1. *For any initial value* $X_0\in\mathbb{R}^7_+$*, the solution* $X(t)=(S(t),Q(t),E(t),A(t),I(t),H(t),R(t))$ *of system (1) verifies the following properties:*

1. $\lim\limits_{t\to\infty}\frac{X_k(t)}{t}=0$ *a.s.* $\forall k\in\{1,2,\dots,7\}$.
2. *Moreover, if* $\mu>\frac{1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\vee\sigma_7^2\right)$*, then*
$\lim\limits_{t\to\infty}\frac{\int_0^t X_k(s)\,\mathrm{d}B_k(s)}{t}=0$ *a.s.* $\forall k\in\{1,2,\dots,7\}$.

Proof. The proof of this lemma is similar in spirit to that of lemmas 2.1 and 2.2 of [8] and therefore it is omitted here. ■

Theorem 2. *Let us denote by* $X(t)=(S(t),Q(t),\dots,R(t))$ *the solution of system* (1) *that starts from a given value* $X_0=(S(0),Q(0),E(0),A(0),I(0),H(0),R(0))\in\mathbb{R}^7_+$*. If* $\mu>\frac{\max_{1\leqslant i\leqslant 7}\sigma_i^2}{2}$ *and* $\min_{3\leqslant i\leqslant 5}\sigma_i^2>6\times(\beta_1 S^o-\mu)$*, with* $S^o=\frac{\Lambda}{\mu}\cdot\frac{\lambda+\mu}{\lambda+q+\mu}$*, then*

$$\limsup_{t\to\infty}\frac{\ln(E(t)+A(t)+I(t))}{t}\leqslant\beta_1 S^o-\mu-\frac{\min_{3\leqslant i\leqslant 5}\sigma_i^2}{6}<0 \ \text{a.s.},$$

which means that the disease will die out exponentially with probability one.

Proof. From Itô's formula and system (1), we have

$$\begin{aligned}\mathrm{d}\ln(E+A+I)=&\left[\frac{1}{E+A+I}\left(\left(\beta_1-\beta_2\frac{I}{b+I}\right)S(I+\theta A)-(\varepsilon_A+\gamma_A+d_A)A-(\varepsilon_I+\gamma_I+d_I)I\right)\right.\\&\left.-\mu-\frac{\sigma_3^2E^2+\sigma_4^2\,A^2+\sigma_5^2\,I^2}{2(E+A+I)^2}\right]\mathrm{d}t+\frac{\sigma_3E\,\mathrm{d}B_3(t)+\sigma_4A\,\mathrm{d}B_4(t)+\sigma_5I\,\mathrm{d}B_5(t)}{E+A+I}.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{d}\ln(E+A+I)\leqslant&\left[\left(\beta_1-\beta_2\frac{I}{b+I}\right)S-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{2}\times\frac{E^2+A^2+I^2}{(E+A+I)^2}\right]\mathrm{d}t\\&+\sigma_3\frac{E}{E+A+I}\,\mathrm{d}B_3(t)+\sigma_4\frac{A}{E+A+I}\,\mathrm{d}B_4(t)+\sigma_5\frac{I}{E+A+I}\,\mathrm{d}B_5(t).\end{aligned}$$

By using the famous Cauchy–Schwartz inequality (see for instance [9] and the references given there), we can assert that $\frac{E^2+A^2+I^2}{(E+A+I)^2}\geqslant\frac{1}{3}$. Hence

$$\begin{aligned}\mathrm{d}\ln(E+A+I)\leqslant&\left[\beta_1S-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}\right]\mathrm{d}t+\sigma_3\frac{E}{E+A+I}\,\mathrm{d}B_3(t)+\sigma_4\frac{A}{E+A+I}\,\mathrm{d}B_4(t)\\&+\sigma_5\frac{I}{E+A+I}\,\mathrm{d}B_5(t),\end{aligned}\tag{7}$$

Integrating (7) from 0 to t, and then dividing by t on both sides, we get

$$\begin{aligned}\frac{\ln(E(t)+A(t)+I(t))}{t}\leqslant&\frac{\ln(E(0)+A(0)+I(0))}{t}+\beta_1\langle S(t)\rangle-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}\\&+\frac{\sigma_3}{t}\int_0^t\frac{E(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_3(s)+\frac{\sigma_4}{t}\int_0^t\frac{A(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_4(s)\\&+\frac{\sigma_5}{t}\int_0^t\frac{I(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_5(s).\end{aligned}\tag{8}$$

On the other hand, the first equation of (1) gives

$$\begin{aligned} S(t)-S(0) &= \Lambda t-\int_0^t\left(\beta_1-\beta_2\frac{I(s)}{b+I(s)}\right)S(s)(I(s)+\theta A(s))\,\mathrm{d}s+\lambda\int_0^t Q(s)\,\mathrm{d}s \\ &\quad-(q+\mu)\int_0^t S(s)\,\mathrm{d}s+\sigma_1\int_0^t S(s)\,\mathrm{d}B_1(s) \\ &\leqslant \Lambda t+\lambda\int_0^t Q(s)\,\mathrm{d}s-(q+\mu)\int_0^t S(s)\,\mathrm{d}s+\sigma_1\int_0^t S(s)\,\mathrm{d}B_1(s). \end{aligned}$$

Therefore

$$\begin{aligned} \langle S(t)\rangle=\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s &\leqslant \frac{1}{q+\mu}\left(\Lambda+\frac{\lambda}{t}\int_0^t Q(s)\,\mathrm{d}s+\frac{S(0)}{t}+\frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s)-\frac{S(t)}{t}\right) \\ &\leqslant \frac{1}{q+\mu}\left(\Lambda+\lambda\langle Q(t)\rangle+\frac{S(0)}{t}+\frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s)\right). \end{aligned} \tag{9}$$

Also, the second one gives

$$Q(t)-Q(0)=q\int_0^t S(s)\,\mathrm{d}s-(\mu+\lambda)\int_0^t Q\,\mathrm{d}s+\sigma_2\int_0^t Q(s)\,\mathrm{d}B_2(s),$$

which shows that

$$\langle Q(t)\rangle=\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}s=\frac{1}{\lambda+\mu}\left(\frac{Q(0)-Q(t)}{t}+\frac{q}{t}\int_0^t S(s)\,\mathrm{d}s+\frac{\sigma_2}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\right) \tag{10}$$

$$\leqslant \frac{1}{\lambda+\mu}\times\frac{Q(0)}{t}+\frac{q}{\lambda+\mu}\langle S(t)\rangle+\frac{\sigma_2}{\lambda+\mu}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s). \tag{11}$$

Combining (9) with (11) yields

$$\begin{aligned} \langle S(t)\rangle \leqslant \frac{1}{q+\mu}&\left(\Lambda+\lambda\left(\frac{Q(0)}{(\lambda+\mu)t}+\frac{q}{\lambda+\mu}\langle S(t)\rangle+\frac{\sigma_2}{\lambda+\mu}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\right)\right. \\ &\left.+\frac{S(0)}{t}+\frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \langle S(t)\rangle &\leqslant \frac{\Lambda(\lambda+\mu)}{\mu(q+\mu+\lambda)}+\frac{\lambda}{\mu(q+\mu+\lambda)}\frac{Q(0)}{t}+\frac{\lambda+\mu}{\mu(q+\mu+\lambda)}\frac{S(0)}{t} \\ &\quad+\frac{\lambda\sigma_2}{\mu(q+\mu+\lambda)}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)+\sigma_1\frac{(\lambda+\mu)}{\mu(q+\mu+\lambda)}\times\frac{1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s). \end{aligned} \tag{12}$$

Since $\mu>\frac{1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\vee\sigma_7^2\right)$, we can conclude by virtue of Lemma 1(2) and inequality (12) that

$$\lim_{t\to\infty}\langle S(t)\rangle\leqslant\frac{\Lambda}{\mu}\times\frac{\lambda+\mu}{q+\mu+\lambda}=S^o. \tag{13}$$

According to the strong law of large numbers for local martingales (see [6, page 12]), we have

$$\begin{cases} \lim\limits_{t\to\infty}\dfrac{\sigma_3}{t}\displaystyle\int_0^t\frac{E(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_3(s)=0 \quad \text{a.s.}, \\ \lim\limits_{t\to\infty}\dfrac{\sigma_4}{t}\displaystyle\int_0^t\frac{A(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_4(s)=0 \quad \text{a.s.}, \\ \lim\limits_{t\to\infty}\dfrac{\sigma_5}{t}\displaystyle\int_0^t\frac{I(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_5(s)=0 \quad \text{a.s.} \end{cases} \tag{14}$$

From (8), (13) and (14) we get

$$\limsup_{t\to\infty}\frac{\ln(E(t)+A(t)+I(t))}{t}\leqslant\beta_1 S^o-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}<0 \quad \text{a.s.},$$

which is exactly the desired conclusion. ■

4. Persistence in the mean of COVID-19

In the following, we give a condition for the persistence in the mean of the disease, but before stating the main result, we shall first recall the concept of persistence in the mean.

Definition 2 (Persistence in the mean [10, 11]). *For system* (1)*, the infectious individuals* $A(t)$ *and* $I(t)$ *are said to be strongly persistent in the mean, or just persistent in the mean, if* $\liminf_{t\to\infty}\langle A(t)+I(t)\rangle > 0$ *almost surely.*

For brevity and simplicity in writing the next results, it will be convenient to adopt the following notations:

- $\rho_1(\alpha) = 3 \times \sqrt[3]{\Lambda(\beta_1-\beta_2)\sigma} \times \left(\sqrt[3]{\theta(1-p)\times\alpha} + \sqrt[3]{p\times(1-\alpha)}\right)$, $\forall\alpha\in(0,1)$,
- $\rho_2 = 7\mu + \sigma + (\varepsilon_A+\gamma_A+d_A) + (\varepsilon_I+\gamma_I+d_I) + (d_H+\gamma_H) + |\lambda-q| + \frac{1}{2}\sum_{i=1}^{7}\sigma_i^2$,
- $\widehat{\alpha} = \frac{\sqrt{\theta(1-p)}}{\sqrt{\theta(1-p)}+\sqrt{p}} \in (0,1)$.

Lemma 2. *For any* $\alpha\in(0,1)$*, the following inequality is satisfied* $\rho_1(\alpha) \leqslant \rho_1(\widehat{\alpha})$*. In other terms,* $\rho_1(\widehat{\alpha})$ *is the maximum value of* $\rho_1(\alpha)$ *on the open interval* $(0,1)$*.*

Proof. We start our proof by observing that the function $\rho_1(\alpha)$ is differentiable on $(0,1)$, with the first derivative given by:

$$\begin{aligned}
\rho_1'(\alpha) := \frac{\mathrm{d}\rho_1(\alpha)}{\mathrm{d}\alpha} &= \sqrt[3]{\Lambda(\beta_1-\beta_2)\sigma}\left(\frac{\sqrt[3]{\theta(1-p)}}{\sqrt[3]{\alpha^2}} - \frac{\sqrt[3]{p}}{\sqrt[3]{(1-\alpha)^2}}\right)\\
&= \frac{\sqrt[3]{\Lambda(\beta_1-\beta_2)\sigma}}{\sqrt[3]{(\alpha\times(1-\alpha))^2}} \times \frac{\theta(1-p)(1-\alpha)^2 - p\alpha^2}{\left(\sqrt[3]{\theta(1-p)(1-\alpha)^2}\right)^2 + \sqrt[3]{\theta(1-p)p(1-\alpha)^2\alpha^2} + \left(\sqrt[3]{p\alpha^2}\right)^2}\\
&= \frac{\sqrt[3]{\Lambda(\beta_1-\beta_2)\sigma}\left(\sqrt{\theta(1-p)}(1-\alpha)+\sqrt{p}\alpha\right)\left(\sqrt{\theta(1-p)}+\sqrt{p}\right)}{\left((1-\alpha)\sqrt[3]{\theta(1-p)\alpha}\right)^2 + \sqrt[3]{\theta p(1-p)\alpha^4(1-\alpha)^4} + \left(\alpha\sqrt[3]{p(1-\alpha)}\right)^2} \times (\widehat{\alpha}-\alpha).
\end{aligned}$$

As it can be seen, the derivative $\rho_1'(\alpha)$ and the linear function $L(\alpha) = \widehat{\alpha}-\alpha$ have the same sign, so the function $\rho_1(\alpha)$ decreases for $\alpha\in(0,\widehat{\alpha})$ and increases for $\alpha\in(\widehat{\alpha},1)$. Therefore, the highest value of ρ_1 in the interval $(0,1)$ is $\rho_1(\widehat{\alpha})$, and this is precisely the assertion of the lemma. ■

Theorem 3. *If* $\rho_1(\widehat{\alpha}) > \rho_2$*, then for any* $X_0\in\mathbb{R}_+^7$*, the solution* $X(t) = (S(t), Q(t), E(t), A(t), I(t), H(t), R(t))$ *of the initial-value problem* (1) *verifies the following property:*

$$\liminf_{t\to\infty}\langle I(t)+A(t)\rangle \geqslant \frac{1}{\beta_1}\left(\rho_1(\widehat{\alpha})-\rho_2\right) > 0 \quad \text{a.s.},$$

which is to say that the infectious individuals $A(t)$ *and* $I(t)$ *are persistent in the mean.*

Proof. Consider the function

$$\begin{array}{rccc}\widehat{V}: & \mathbb{R}_+^7 & \longrightarrow & \mathbb{R}\\ & x & \longmapsto & \sum_{i=1}^{7}\ln x_i.\end{array}$$

From Itô's formula and system (1), we have

$$\begin{aligned}
\mathrm{d}\widehat{V}(X(t)) = \Bigg(&\left[\frac{\Lambda}{S} - \left(\beta_1-\beta_2\frac{I}{b+I}\right)(I+\theta A) + \lambda\frac{Q}{S} - (q+\mu)\right] + \left[q\frac{S}{Q} - (\lambda+\mu)\right]\\
&+ \left[\left(\beta_1-\beta_2\frac{I}{b+I}\right)\frac{S}{E}(I+\theta A) - (\mu+\sigma)\right] + \left[(1-p)\sigma\frac{E}{A} - (\mu+\varepsilon_A+\gamma_A+d_A)\right]\\
&+ \left[\sigma p\frac{E}{I} - (\mu+\varepsilon_I+\gamma_I+d_I)\right] + \left[\varepsilon_I\frac{I}{H} + \varepsilon_A\frac{A}{H} - (d_H+\gamma_H+\mu)\right]\\
&+ \left[\gamma_H\frac{H}{R} + \gamma_I\frac{I}{R} + \gamma_A\frac{A}{R} - \mu\right] - \frac{1}{2}\sum_{i=1}^{7}\sigma_i^2\Bigg)\,\mathrm{d}t + \sum_{i=1}^{7}\sigma_i\,\mathrm{d}B_i(t)
\end{aligned}$$

$$\geqslant \left(\frac{\Lambda}{S} - \beta_1(I+\theta A) + (\lambda \wedge q)\left(\frac{S}{Q} + \frac{Q}{S}\right) + (\beta_1 - \beta_2)\frac{S}{E}(I+\theta A) + (1-p)\sigma\frac{E}{A} + \sigma p\frac{E}{I}\right.$$
$$\left. - \left[7\mu + \lambda + q + \sigma + (\varepsilon_A + \gamma_A + d_A) + (\varepsilon_I + \gamma_I + d_I) + (d_H + \gamma_H) + \frac{1}{2}\sum_{i=1}^{7}\sigma_i^2\right]\right)\mathrm{d}t$$
$$+ \sum_{i=1}^{7}\sigma_i \, \mathrm{d}B_i(t).$$

Noticing that $\lambda \wedge q = \frac{\lambda+q-|\lambda-q|}{2}$ and $\left(\frac{S}{Q} + \frac{Q}{S}\right) \geqslant 2$, we get for all $t \geqslant 0$

$$\mathrm{d}\widehat{V}(X(t)) \geqslant \left(\left[\frac{(1-\widehat{\alpha})\Lambda}{S} + (\beta_1-\beta_2)\frac{SI}{E} + \sigma p\frac{E}{I}\right] + \left[\frac{\widehat{\alpha}\Lambda}{S} + \theta(\beta_1-\beta_2)\frac{SA}{E} + (1-p)\sigma\frac{E}{A}\right]\right.$$
$$\left. - \beta_1(I+\theta A) - \rho_2\right)\mathrm{d}t + \sum_{i=1}^{7}\sigma_i \, \mathrm{d}B_i(t),$$

and from the relation between arithmetic and geometric means (the first is greater than or equal to the second, see [12]), it results that

$$\mathrm{d}\widehat{V}(X(t)) \geqslant \left(3 \times \sqrt[3]{(1-\widehat{\alpha})\Lambda(\beta_1-\beta_2)\sigma p} + 3 \times \sqrt[3]{\widehat{\alpha}\Lambda(\beta_1-\beta_2)\theta\sigma(1-p)} - \beta_1(I+\theta A) - \rho_2\right)\mathrm{d}t$$
$$+ \sum_{i=1}^{7}\sigma_i \, \mathrm{d}B_i(t)$$
$$\geqslant \left(\left(\rho_1(\widehat{\alpha}) - \rho_2\right) - \beta_1(I+\theta A)\right)\mathrm{d}t + \sum_{i=1}^{7}\sigma_i \, \mathrm{d}B_i(t). \qquad (15)$$

Integrating from 0 to t and dividing by t on both sides of (15) gives

$$\frac{\widehat{V}(X(t)) - \widehat{V}(X(0))}{t} \geqslant \left(\rho_1(\widehat{\alpha}) - \rho_2\right) - \beta_1\langle I(t) + \theta A(t)\rangle + \sum_{i=1}^{7}\sigma_i\frac{B_i(t)}{t}.$$

Hence

$$\langle I(t) + A(t)\rangle \geqslant \langle I(t) + \theta A(t)\rangle \geqslant \frac{1}{\beta_1}\left(\frac{\widehat{V}(X(0)) - \widehat{V}(X(t))}{t} + \left(\rho_1(\widehat{\alpha}) - \rho_2\right)\right) + \sum_{i=1}^{7}\frac{\sigma_i}{\beta_1}\frac{B_i(t)}{t}. \qquad (16)$$

Since $\ln y \leqslant y - 1 \leqslant y$ for all $y > 0$, one can assert that $\widehat{V}(x) \leqslant \sum_{i=1}^{7} x_i$ for any $x \in \mathbb{R}^7_+$.

Combining the last inequality with (16) yields

$$\langle I(t) + A(t)\rangle \geqslant \frac{1}{\beta_1}\left(\frac{\widehat{V}(X(0))}{t} - \frac{1}{t}\sum_{i=1}^{7}X_i(t) + \left(\rho_1(\widehat{\alpha}) - \rho_2\right)\right) + \sum_{i=1}^{7}\frac{\sigma_i}{\beta_1}\frac{B_i(t)}{t}.$$

By using the strong law of large numbers for local martingales and the first assertion of Lemma 1, we obtain

$$\liminf_{t\to\infty}\langle I(t) + A(t)\rangle \geqslant \frac{1}{\beta_1}\left(\rho_1(\widehat{\alpha}) - \rho_2\right) > 0 \quad \text{a.s.},$$

which is the required assertion. ■

Remark 1. In the last proof, we can notice that any constant $\alpha \in (0,1)$ can play the role of $\widehat{\alpha}$, but the peculiarity of the latter lies essentially in its capacity to weaken the hypothesis of Theorem 3. Indeed, according to Lemma 2, if $\rho_1(\alpha) > \rho_2$ for some $\alpha \in (0,1)$ then necessarily $\rho_1(\widehat{\alpha}) > \rho_2$.

5. Numerical simulation examples

In this section, using the parameter values as shown in Table 1, we present some numerical simulations to validate the various results proved in this paper. Most of the parametric values appearing in this table (Table 1) are selected from real data available in existing literature (Refs. [1,13–15] more precisely) and the rest of them are just assumed for numerical calculations. The solution of our stochastic COVID-19 model, is simulated in our case with the initial state given by $S(0) = 1.8 \times 10^6$, $Q(0) = 0$, $E(0) = 10$,

$A(0) = 15$, $I(0) = 8$, $H(0) = 5$ and $R(0) = 0$ (see [13]). In what follows, the unity of time is one day and the number of individuals is expressed in one million population.

Table 1. Definitions and values (per day) of COVID-19 model parameters used in the simulation.

Parameter	Description	Nominal value
Λ	Recruitment rate	108.63
β_1	Contact rate in absence of media coverage	$(1.7 \times 10^{-9}, 5.2 \times 10^{-3})$
β_2	Awareness rate (or also response intensity)	$[0, \beta_1]$
b	Constant of media's half saturation	70
θ	Modification ratio of asymptomatic infectiousness	0.0494
q	Quarantine rate	0.071
λ	Rate of release from quarantine	0.1003
μ	Natural death rate	0.00029
σ	The transition rate of exposed individuals to the infective classes	0.2
p	Probability of having symptoms among infected individuals	$(0, 1)$
ε_A	The hospitalization rate of asymptomatic infected individuals	0.1
γ_A	Recovery rate of asymptomatic infected individuals	0.15
d_A	Disease-induced death rate for asymptomatic infected individuals	0.005
ε_I	The hospitalization rate of symptomatic infected individuals	0.33
γ_I	Recovery rate of symptomatic infected individuals	0.1001
d_I	Disease-induced death rate for symptomatic infected individuals	0.008
γ_H	Recovery rate of hospitalized individuals	0.14
d_H	Disease-induced death rate for hospitalized individuals	0.004

Example 1 (Asymptotic behavior). In order to exhibit the random fluctuations effect on COVID-19 dynamics, we present in Figures 1 and 2 a collection of numerical simulations. In the first instance, we take $\beta_1 = 2.08 \times 10^{-9}$, $\beta_2 = 0.6 \times \beta_1$, $p = 0.6201$, and we choose the stochastic intensities as follows: $\sigma_1 = 0.024$, $\sigma_2 = 0.0235$, $\sigma_3 = 0.015$, $\sigma_4 = 0.0174$, $\sigma_5 = 0.019$, $\sigma_6 = 0.0213$, and $\sigma_7 = 0.0238$. Then,

$$\frac{1}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2\right) = 0.000288 < 0.000290 = \mu,$$

and $\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2 = 0.000225 > 0.000202 = 6(\beta_1 S^o - \mu)$. Hence, the assumptions of Theorem 2 are verified, and consequently

$$\limsup_{t\to\infty} \frac{\ln(E(t) + A(t) + I(t))}{t} \leqslant \beta_1 S^o - \mu - \frac{\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2}{6} = -3.83 \times 10^{-6} < 0 \ \text{ a.s.}$$

That is to say that the COVID-19 dies out exponentially almost surely. The last result is confirmed by the curves depicted in Figure 1. To make the condition $\rho_1(\widehat{\alpha}) > \rho_2$ true, we take $\beta_1 = 4.1 \times 10^{-3}$, $\beta_2 = 0.1 \times \beta_1$ and we select new values of stochastic intensities as follows: $\sigma_1 = 0.019$, $\sigma_2 = 0.0185$, $\sigma_3 = 0.014$, $\sigma_4 = 0.017$, $\sigma_5 = 0.0158$, $\sigma_6 = 0.0136$, and $\sigma_7 = 0.0182$. Thus, the main result of Theorem 3 is satisfied and this time, the COVID-19 persists in the mean as shown in Figure 2.

Example 2 (The effectiveness of media intervention and quarantine strategies). We aim during this example to examine numerically the impact of media intrusion and quarantine strategies on the COVID-19 spread. To this end, we simulate the progression of the total infected population number with various values of β_2, λ and q. Through Figure 3, we can perceive that the increase of the quarantine rate and duration can delay the arrival of infection peak, reduce remarkably the impact of the disease, and even lead it to the extinction sometimes (see for example the last two curves presented in Figure 3). On the other hand, and as it can be seen from Figures 3 and 4, the media alert strategy is able also to diminish the severity of the COVID-19 spread, but it can not make it disappear, and we explain this theoretically by the absence of the parameters β_2 and b in the persistence and extinction conditions (for example $\mathcal{R}_0$ does not involve these parameters). Roughly speaking, the role of the quarantine and the information intervention about COVID-19 is critically important, particularly in its beginnings. The growth of the positive response in susceptible individuals leads to reduce the gravity of the infection and creates a conscious public able to overcome this new pandemic by respecting social distancing and self-isolation procedures.

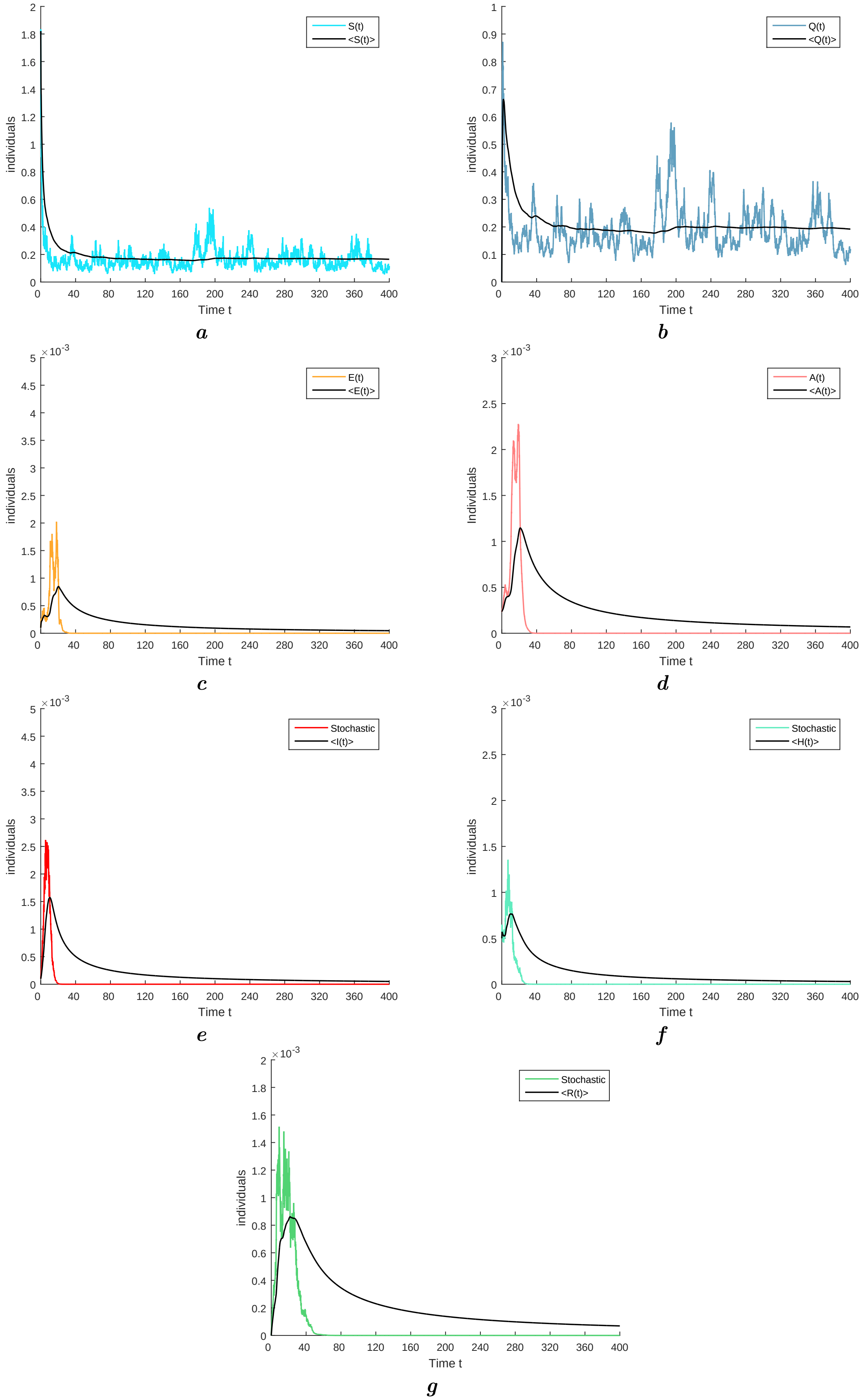

Fig. 1. Trajectories of COVID-19 stochastic model (1) taking $\beta_1 = 2.08 \times 10^{-9}$, $\beta_2 = 0.6 \times \beta_1$ and $p = 0.6201$.

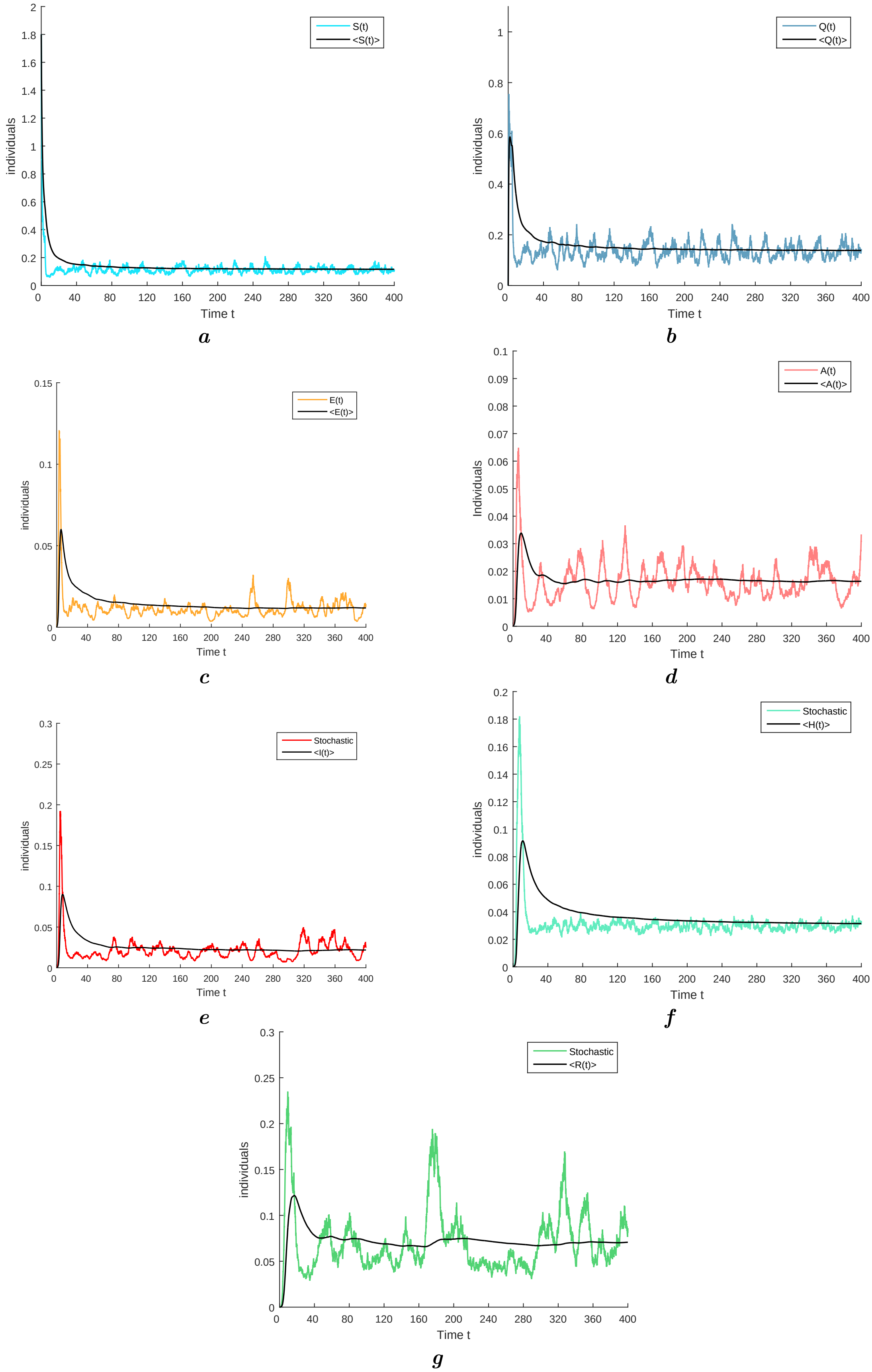

Fig. 2. Trajectories of COVID-19 stochastic model (1) taking $\beta_1 = 4.1 \times 10^{-3}$, $\beta_2 = 0.1 \times \beta_1$ and $p = 0.6201$ $(\rho_1(\widehat{\alpha}) = 1.0266 > 0.9694 = \rho_2)$.

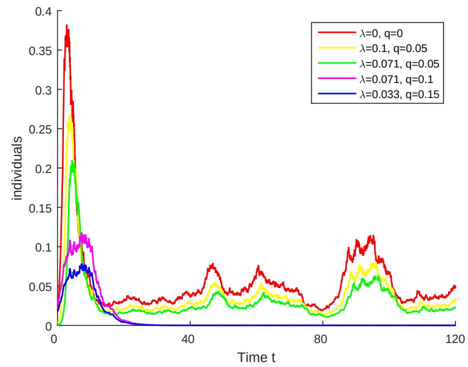

Fig. 3. The impact of the quarantine parameters λ and q on the trajectories of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$. The rest of the parameters is taken respectively as in Figures 1 and 2.

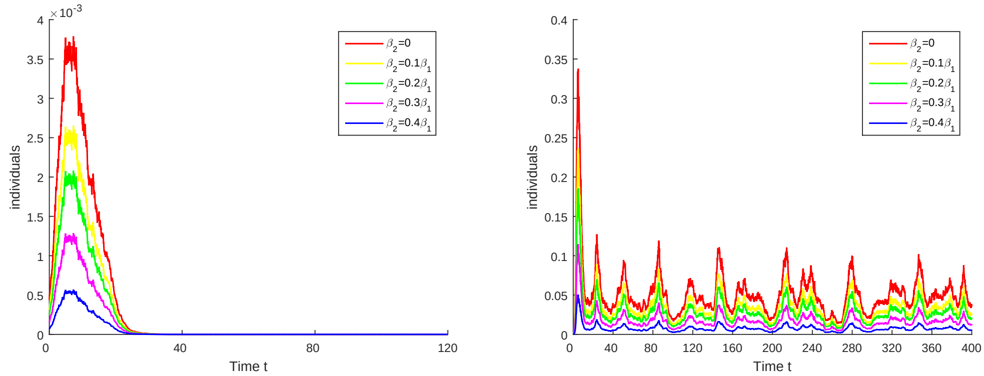

Fig. 4. Stochastic paths of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$ under different response intensities β_2. The other parameters are taken respectively as in Figures 1 and 2.

6. Conclusion

In this study, we have analyzed and explored the COVID-19 perturbed system (1). First, we have demonstrated the existence and uniqueness of a global positive solution to this system. Then, we have derived the conditions for COVID-19 extinction and persistence, and we remarked that they are mainly depending on the magnitude of the noises intensities as well as the system parameters. Finally, we have presented some numerical simulation examples to support and visualize our findings.

[1] Jia J., Ding J., Liu S., Liao G., Li J., Duan B., Wang G., Zhang R. Modeling the control of COVID-19: Impact of policy interventions and meteorological factors. Preprint arXiv:2003.02985 (2020).

[2] Pawar D. D., Patil W. D., Raut D. K. Fractional-order mathematical model for analysing impact of quarantine on transmission of COVID-19 in India. Mathematical Modeling and Computing. **8** (2), 253–266 (2021).

[3] Ilnytskyi J. M. Modeling of the COVID-19 pandemic in the limit of no acquired immunity. Mathematical Modeling and Computing. **8** (2), 282–303 (2021).

[4] Yavorska O., Bun R. Spatial analysis of COVID-19 spread in Europe using "center of gravity" concept. Mathematical Modeling and Computing. **9** (1), 130–142 (2022).

[5] Kouidere A., Elhia M., Balatif O. A spatiotemporal spread of COVID-19 pandemic with vaccination optimal control strategy: A case study in Morocco. Mathematical Modeling and Computing. **10** (1), 171–185 (2023).

[6] Mao X. Stochastic differential equations and applications. Woodhead Publishing (2007).

[7] Karatzas I., Shreve S. E. Brownian Motion and Stochastic Calculus. Springer New York, NY (1998).

[8] Zhao Y., Jiang D. The threshold of a stochastic SIS epidemic model with vaccination. Applied Mathematics and Computation. **243**, 718–727 (2014).

[9] Yin S. A New Generalization on Cauchy–Schwarz Inequality. Journal of Function Spaces. **2017**, 9576375 (2017).

[10] Song Y., Miao A., Zhang T., Wang X., Liu J. Extinction and persistence of a stochastic SIRS epidemic model with saturated incidence rate and transfer from infectious to susceptible. Advances in Difference Equations. **2018**, 293 (2018).

[11] Sun F. Dynamics of an imprecise stochastic Holling II one-predator two-prey system with jumps. Preprint arXiv:2006.14943 (2020).

[12] Nicholson J., Clapham C. The Concise Oxford Dictionary of Mathematics. Vol. 5. Oxford University Press Oxford (2014).

[13] Tang B., Wang X., Li Q., Bragazzi N. L., Tang S., Xiao Y., Wu J. Estimation of the transmission risk of the 2019-nCoV and its implication for public health interventions. Journal of Clinical Medicine. **9** (2), 462 (2020).

[14] Wu J., Tang B., Bragazzi N. L., Nah K., McCarthy Z. Quantifying the role of social distancing, personal protection and case detection in mitigating COVID-19 outbreak in Ontario, Canada. Journal of Mathematics in Industry. **10**, 15 (2020).

[15] Public Health Ontario. Ontario COVID-19 Data Tool. PHO official website (2020).

Математичне дослідження розповсюдження COVID-19 через стохастичну модель епідемії

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COVID-19 є великою небезпекою, яка загрожує всьому світу. У цьому контексті математичне моделювання є дуже потужним інструментом, щоб дізнатися більше про те, як така хвороба передається всередині людської популяції. У зв'язку з цим у цій статті пропонується стохастична модель епідемії, яка описує динаміку COVID-19 під час застосування карантину та стратегій медіа-висвітлення, і здійснено строгий математичний аналіз цієї моделі, щоб отримати загальне уявлення про поширення COVID-19.

Ключові слова: *COVID-19; броунівський рух; стохастична модель епідемії; висвітлення ЗМІ; карантин.*