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Numerical modeling of heat and mass transfer processes in a capillary-porous body during contact drying

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The problem of conductive (contact) drying of a capillary-porous body in a steam-air (gas) environment by heat transfer to the material during its contact with the heated surfaces of the material is considered. A system of significantly nonlinear differential equations of heat and mass transfer to describe such a process is obtained. To solve the formulated problem of heat and mass transfer (without taking into account deformability), the method of solving nonlinear boundary value problems is applied in the form of an iterative process, at each step of which a linear boundary value problem is solved. The results of the application of the method are verified based on the popular numerical scheme used. They agree well. A numerical experiment is conducted for materials of three types of porosity. The results are presented graphically and tabularly. The regularities of contact drying of capillary-porous materials in a steam-air environment are deduced.

Keywords: *contact drying; capillary-porous materia; system of nonlinear differential equations; iterative process; linear boundary value problem.*

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1. Introduction

When studying the processes of heat and mass transfer in porous bodies, sufficiently universal continuum-thermodynamic approaches of continuum mechanics are used, which are based on the methods of thermodynamics of non-equilibrium processes, as well as new approaches are being developing actively. Thus, the non-equilibrium thermodynamic relations for the non-equilibrium one-particle distribution function of particles are obtained in [1]. A kinetic approach based on a modified chain of BBGKI equations for nonequilibrium particle distribution functions is used in [2] to describe the ion transfer processes in the ionic solution–porous medium system. The issue of accounting for the temperature inhomogeneity in the model of heat transfer for the structure being attributed to the heating-cooling cycles, or nonuniform heating, is considered in [3]. Stefan's linear problem for drying capillary-porous material is solved under quasi-averaged formulation in [4]. In [5], the problem of determination of the coefficients of internal diffusion of moisture for capillary-porous materials of plant origin during filtration drying is solved based on integral transformations. The problem of mutual phase distribution in investigation of drying the capillary-porous solid of a cylindrical shape in [6] is solved using the principle of local phase equilibrium. The algorithm for numerical analysis of the mathematical model of coupled heat and mass transfer problems of drying material in pulsed drying mode is suggested in [7]. The diffusion of heat in the material is described by the generalized version of the Cattaneo–Maxwell diffusion equation [8]. A complex algorithm comprising the specific mechanisms of drying in the first and second periods of drying is constructed in [9]. In [10], there are adopted the long short-term memory neural network, backpropagation neural network, and Central–Composite response surface method to establish a moisture content prediction model and a process parameter optimization model based on single-factor experiments. The model of drying capillary porous materials, such as fruits and vegetables, which takes into account simultaneous heat and mass transport with anisotropic deformation, is developed in [11].

When performing structural analysis of conductive (contact) drying in a steam-air (gas) environment when heat is transferred under its contact with the material, researchers often face the difficulties with solving the governing equations describing models, which are significantly nonlinear [12]. As a method of solving non-linear contact-boundary value problems, we suggest applying an iterative method, at each step of which a linear boundary value problem will be solved. Deformability is not taken into account.

2. The problem formulation and a system of basic equations

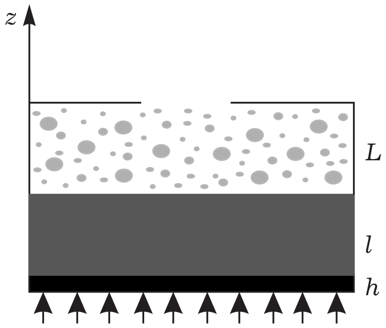

Fig. 1. Schematic representation of the model.

Let us consider the following problem.

Consider a thin plane plate, a surface of which from one side is subjected to the external heat flow $q_e(t)$ (Figure 1). Conductive contact drying takes place in a steam-air (gas) environment by transferring heat to the material when it is in contact with heated surfaces. The plate has an area S, thickness h_w, its material is characterized by density ρ_w, specific heat capacity C_w. A layer of capillary-porous moisture-saturated material of the thickness l is placed on this plate. The capillary-porous material has the porosity Π, density ρ_0^0, specific heat capacity C_0, and thermal conductivity coefficient in the dry state λ_0.

From the open side of the capillary-porous material, the moisture evaporates into the cavity of the volume V and the depth $L = V/S$. There is an outlet in the cavity through which the steam-air mixture flows into the environment under pressure P_e. The cavity is thermally insulated. We neglect the heat capacity of its walls. Such an installation can serve as an example of a drying chamber for conductive drying.

The system of heat and mass transfer equations is described as follows:

$$\frac{T}{T_0}\frac{d}{d\tau}\left[\rho_0 C_{ef}T + r_0\Pi(1-\alpha)\rho_v^0\right] = \nabla\Big[\lambda_{ef}\nabla T - r_0 J_v - \sum_{i=l,v,a} C_{pi}TJ_i\Big], \tag{1}$$

$$P_g = \left(\frac{\rho_v^0}{M_v} + \frac{\rho_a^0}{M_a}\right)RT,$$

$$\Pi\frac{\partial\left[\left(\rho_L^0-\rho_v^0\right)\alpha + \rho_v^0\right]}{\partial\tau} + \nabla J_m = 0, \tag{2}$$

$$J_L + J_v = J_m,$$

$$\Pi\frac{\partial\left[(1-\alpha)\rho_a^0\right]}{\partial\tau} + \boldsymbol{\nabla} J_a = 0, \tag{3}$$

$$\rho_v^0 = 133\frac{M_v}{RT}\exp\left(18.681 - \frac{4105}{T-35}\right). \tag{4}$$

Π, α, r_0, λ_{ef}, J_i are the porosity, relative moisture saturation, specific heat of vaporization, effective thermal conductivity, and moisture, steam, and air flows, respectively. If the evaporation is not strong, then it can be roughly assumed that the steam pressure in the cavity is equal to the saturation pressure. In this system of equations, the temperature T, moisture saturation α, and air density ρ_a^0 are unknown. At the initial moment, there can be moisture, air, steam in the pores. We assume that the steam-air mixture is a mixture of ideal gases and in the wet state, when the capillary-porous material is saturated with moisture $\alpha > 0$, the density of steam-air mixture is a function of temperature only. The equation does not include the phase transition criterion, the dependence of which on the parameters is complex. The equations remain valid in the dry zone, where there is no moisture, and $\alpha = 0$, $I_v = 0$ in this domain, Eq. (3) serves to determine the moisture density.

The boundary conditions are formulated as follows: at the initial moment of time, the pressure of the steam-air mixture in the capillary-porous material and in the cavity is equal to the external atmospheric pressure P_e: $P_g = P_e(0) = P_0$.

The initial temperature

$$T(x,0) = T_0. \tag{5}$$

The moisture saturation $\alpha(x,0) = \alpha_0 \leqslant 1$.

The air density $\rho_a^0(x,0) = \frac{P_0 - P_{vs}}{RT_0} M_a$.

The boundary conditions on the side of the heated plate are as follows:

$$q_e = -\lambda \nabla T + \rho_w C_w h_w \frac{\partial T}{\partial \tau} + r^* J_v, \quad r^*(T) = r_0 - (C_L - C_{pv})\, T. \tag{6}$$

The moisture and air flows at the interface from the side of the plate are zero:

$$J_m = 0, \quad J_a = 0.$$

The boundary conditions on the surface of the capillary-porous material from the side of the cavity with the opening for $x = l$ are as follows:

$$-\lambda \nabla T = r^* J_L + \frac{V}{S}\left(\rho_a^0 C_{va} + \rho_v^0 C_{vv}\right)\frac{\partial T}{\partial t} + \frac{V}{S}\left[R_a \frac{\partial \rho_a^0}{\partial t} + R_v \frac{\partial \rho_v^0}{\partial t}\right] T, \tag{7}$$

where the first term $\lambda \nabla T$ characterizes the heat flow that penetrates inside the body; the second term is equal to the product of the specific heat of vaporization multiplied by the density of the moisture flow that evaporates; the third term is the power spent on heating the surface; the fourth term is the flow of heat transmitted by the movement of the steam-air mixture.

The total flow of vaporized moisture should be equal to the flow rate of the moisture flowing out through the hole, to estimate which we will use the formula for adiabatic output from the cavity [13]. To determine the flow of moisture, the equation of conservation of moisture mass in the cavity is used

$$S J_m = Q_e \frac{\rho_v^0}{\rho_g^0} + V \frac{\partial \rho_v^0}{\partial t}, \tag{8}$$

the air flow:

$$S J_a = Q_e \frac{\rho_a^0}{\rho_g^0} + V \frac{\partial \rho_a^0}{\partial t}, \quad \rho_v^0 = \rho_{vs}(T_c). \tag{9}$$

The vapor density is equal to the saturated vapor density. The movement of gas in the cavity into which evaporation occurs is neglected. The gas temperature in the cavity is assumed to be the same throughout the volume.

The flow of the steam-air mixture through the drainage hole is determined by the formulae of output from the cavity

$$Q_e = s\left[\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{1/2} Q(\varepsilon_p)\, P_c (R_g T_c)^{-1/2}, \quad \varepsilon_p = \frac{P_e}{P_c}, \tag{10}$$

$$Q_e Q(\varepsilon_p) = \begin{cases} \left(\dfrac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} \varepsilon_p^{1/\gamma}\left[\dfrac{\gamma+1}{\gamma-1}\left(1 - \varepsilon_p^{\frac{\gamma-1}{\gamma}}\right)\right]^{1/2}, & \varepsilon_p > \left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}, \\ 1, & \varepsilon_p \leqslant \left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}. \end{cases}$$

Here T_c, P_c are temperature and pressure in the cavity, Q_e is the gas flow through the drainage hole [14], γ is the adiabatic index, R_g is a gas constant. The boundary conditions are obtained under the assumption that the gradients of temperature, pressure, and concentration across the cavity are negligible, and the vapor pressure in the cavity is close to the saturation pressure for the cavity temperature.

Let us write the system of nonlinear differential Eqs. (1)–(3) in a matrix form

$$\frac{\partial}{\partial T}[\boldsymbol{E}(u)] + \frac{\partial}{\partial x}\boldsymbol{J} = 0, \tag{11}$$

where $u = (T, \alpha, \rho_a^0)$; $\boldsymbol{E}$ is a vector, the components of which are the total content of enthalpy, moisture, and air in a unit volume of the material; $\boldsymbol{J}$ is a vector composed of heat, moisture, and air flows, it is linearly related to the gradients T, α, P; $c_a = \rho_a^0/\rho_g^0$; $\boldsymbol{J}(u) = -A(u)\frac{\partial \boldsymbol{F}(u)}{\partial x}$; $\boldsymbol{F} = \boldsymbol{F}(T, \alpha, P, c_a)$; $A(u)$ is the 3×4 matrix, and the 3×5 matrix (if capillary pressure is taken into account) [15–17]

$$A = [a_{ij}], \quad i = \overline{1,4}, \quad j = \overline{1,3},$$

where $a_{11} = \lambda$; $a_{12} = 0$; $a_{13} = \frac{\tilde{K}}{\mu_{eff}}(1-\alpha)\big(r_0\rho_v^0 + TC_{pa}\rho_a^0 + TC_{pv}\rho_v^0\big)$;

$$a_{14} = \Pi D_{ef}(1-\alpha)\left(\rho_v^0 + \rho_a^0\right)\left(r_0 - C_{pa}T - C_{pv}T\right); \quad a_{21} = a_L\rho_0\delta; \quad a_{22} = D_L\Pi\rho_L^0;$$

$$a_{23} = \alpha\frac{\tilde{K}\rho_L^0}{\mu_L} + \frac{\tilde{K}\,(1-\alpha)}{\mu_{ef}}\rho_v^0; \quad a_{24} = \Pi D_{ef}(1-\alpha)\left(\rho_v^0 + \rho_a^0\right); \quad a_{31} = 0;$$

$$a_{32} = 0; \quad a_{33} = \frac{\tilde{K}(1-\alpha)}{\mu_{ef}}\rho_a^0; \quad a_{34} = -\Pi D_{ef}(1-\alpha)\left(\rho_v^0 + \rho_a^0\right).$$

Here D_{ef}, C_{pa}, C_{pv} are the coefficients of effective diffusion, specific heat capacities of air and steam at a constant pressure, respectively; λ is the coefficient of effective thermal conductivity,

$$\boldsymbol{E}\big(T, \alpha, \rho_a^0\big) = \begin{bmatrix} T\left\{\rho_0C_0 + \Pi\left[\rho_L^0C_L\alpha + (1-\alpha)\left(\rho_v^0C_{vv} + \rho_a^0C_{va}\right)\right] + r_0\Pi(1-\alpha)\rho_v^0\right\} \\ \Pi\left[\left(\rho_L^0 - \rho_v^0\right)\alpha + \rho_v^0\right] \\ \Pi\,(1-\alpha)\,\rho_v^0 \end{bmatrix};$$

$$A = \begin{bmatrix} \lambda & 0 & \frac{\tilde{K}K_g}{\mu_{ef}}\left(r_0\rho_v^0 + T(C_{pa}\rho_a^0 + C_{pv}\rho_v^0)\right) & \Pi D_{ef}(1-\alpha)\rho_g^0\left(r_0 - (C_{pa} + C_{pv})T\right) \\ D_L\rho_0\delta & D_L\rho_L^0\Pi & \left[\frac{\tilde{K}K_L\rho_L^0}{\mu_L} + \frac{\tilde{K}K_g\rho_v^0}{\mu_{ef}}\right] & \Pi D_{ef}\,(1-\alpha)\left(\rho_v^0 + \rho_a^0\right) \\ 0 & 0 & \frac{\tilde{K}K_g\rho_a^0}{\mu_{ef}} & -\Pi D_{ef}(1-\alpha)\left(\rho_v^0 + \rho_a^0\right) \end{bmatrix};$$

$K_g = 1 - \alpha$ is a relative gas permeability; r_0 is the heat of vaporization for $T = 0\,\mathrm{K}$;

$$\boldsymbol{F} = \begin{bmatrix} T \\ \alpha \\ \left(\frac{\rho_v^0}{M_v} + \frac{\rho_a^0}{M_a}\right)RT \\ \frac{\rho_v^0}{\rho_v^0 + \rho_a^0} \end{bmatrix}.$$

The system of equations have to satisfy the boundary conditions

$$\boldsymbol{J}\big|_{x=0}(u) = \boldsymbol{Q}_0, \quad \boldsymbol{J}\big|_{x=1}(u) = \boldsymbol{Q}_1$$

and the initial conditions: $(0 \leqslant x \leqslant l,\ t = 0)$, $T = T_0$, $\alpha = \alpha_0 < 1$,

$$\rho_a^0 = \left[P_0 - P_{vs}(T)\right]\frac{M_a}{RT_0}, \tag{12}$$

$\boldsymbol{Q}_0 = \begin{bmatrix} q_e - \rho_pC_ph\frac{\partial T}{\partial t} \\ 0 \\ 0 \end{bmatrix}$ is the gas flow from the side of the plate,

$$\boldsymbol{Q}_1 = \begin{bmatrix} \frac{V}{S}\left\{\left(\frac{\partial\rho_a^0}{\partial t}C_{va} + \frac{\partial\rho_v^0}{\partial T}\frac{\partial T}{\partial t}C_{vv}\right)T + \left(\rho_a^0C_{va} + \rho_v^0C_{vv}\right)\frac{\partial T}{\partial t} + r_0\frac{\partial\rho_v^0}{\partial T}\frac{\partial T}{\partial t}\right\} + \frac{Q_e}{S}\frac{\left(\rho_a^0C_{va} + \rho_v^0C_{vv}\right)T + r_0\rho_v^0}{\rho_a^0 + \rho_v^0} \\ \frac{1}{S}\frac{Q_e\rho_v^0}{\rho_a^0 + \rho_v^0} + \frac{V}{S}\frac{\partial\rho_v^0}{\partial T}\frac{\partial T}{\partial t} \\ \frac{1}{S}\frac{Q_e\rho_v^0}{\rho_a^0 + \rho_v^0} + \frac{V}{S}\frac{\partial\rho_a^0}{\partial T} \end{bmatrix}.$$

3. The method of solving the nonlinear problem

We solve the formulated nonlinear boundary value problem by two methods for comparing the accuracy of our results.

Construction of a difference scheme.

Let us integrate the matrix Eq. (11) with respect to x over the interval $x_n - \frac{\Delta x}{2}$, $x_n + \frac{\Delta x}{2}$ for $t = t_k = k\Delta t$. We obtain

$$-\int_{x_n+\frac{\Delta x}{2}}^{x_n-\frac{\Delta x}{2}} \frac{\partial E^k}{\partial t}\, dx + J^k_{n+\frac{1}{2}} - J^k_{n-\frac{1}{2}} = 0. \tag{13}$$

Equation (13) after the difference approximation is reduced to the difference scheme. A three-point approximation of the spatial variables is used. The system of nonlinear algebraic equations is solved by Newton's method.

The second solution method is a linearization method.

In order to solve the boundary value problem, in addition, an iterative process is built, at each step of which a linear boundary value problem is solved for the next approximation, which uses the information of the previous one. A small time step is used to ensure convergence of iterations.

If an approximation $u_i = (T, \alpha, \rho_a^0)$ of the problem solution is known, then the exact solution u^* can be presented as follows $u^* = u_i + \Delta u_i^*$.

Put $u^* = u_{i+1}$.

Based on the Lagrange formula

$$\begin{aligned}\dot{\boldsymbol{E}}\,[u_{i+1}(t,x)] &= \dot{\boldsymbol{E}}\,[u_i(t,x)] + \Big\{ \big[\dot{E}[u_i(t,x)]\big]'_T, \big[\dot{E}[u_i(t,x)]\big]'_\alpha, \big[\dot{E}[u_i(t,x)]\big]'_{\rho_a} \Big\} \begin{bmatrix} T_{i+1}-T_i \\ \alpha_{i+1}-\alpha_i \\ \rho_{a\,i+1}-\rho_{a\,i} \end{bmatrix} \\ &\quad + \Big\{ \big[\dot{E}[u_i(t,x)]\big]'_{\dot{T}}, \big[\dot{E}[u_i(t,x)]\big]'_{\dot{\alpha}}, \big[\dot{E}[u_i(t,x)]\big]'_{\dot{\rho}_a} \Big\} \begin{bmatrix} \dot{T}_{i+1}-\dot{T}_i \\ \dot{\alpha}_{i+1}-\dot{\alpha}_i \\ \dot{\rho}_{a\,i+1}-\dot{\rho}_{a\,i} \end{bmatrix}; \end{aligned} \tag{14}$$

$$\begin{aligned}\boldsymbol{J}(t,x,u_{i+1}) &= -A(t,x,u_{i+1})\frac{\partial \boldsymbol{F}}{\partial x}(t,x,u_{i+1},\dot{u}_{i+1}) \\ &= \boldsymbol{J}[u_i(t,x)] + \big\{ J[u_i(t,x)]'_T, J[u_i(t,x)]'_\alpha, J[u_i(t,x)]'_{\rho_a} \big\} \begin{bmatrix} T_{i+1}-T_i \\ \alpha_{i+1}-\alpha_i \\ \rho_{a\,i+1}-\rho_{a\,i} \end{bmatrix} \\ &\quad + \Big\{ J[u_i(t,x)]'_{T'}, J[u_i(t,x)]'_{\alpha'}, J[u_i(t,x)]'_{\rho'_a} \Big\} \begin{bmatrix} T'_{i+1}-T'_i \\ \alpha'_{i+1}-\alpha'_i \\ \rho'_{a\,i+1}-\rho'_{a\,i} \end{bmatrix}. \end{aligned} \tag{15}$$

Using quadrature formulae of the interpolation type according to the 3/8 rule [18], we obtain the difference scheme

$$\frac{1}{8}\left(\frac{\Delta E^k_{n-1}}{\Delta t} + 6\frac{\Delta E^k_n}{\Delta t} + \frac{\Delta E^k_{n+1}}{\Delta t} \right) + \frac{1}{\Delta x}\big(J^k_{n+1/2} - J^k_{n-1/2}\big) = 0, \tag{16}$$

where $\frac{\Delta E^k_n}{\Delta t} = \frac{1}{\Delta t}\left[E^k_n - E^{k-1}_n\right]$, $E^k_n = E(u^k_n)$,

$$\begin{aligned} \{J\}'_{T/\alpha/\rho_a} &= -\left\{ [A]'_{T/\alpha/\rho_a}\left[\frac{\partial F}{\partial x}\right] + [A]\left[\frac{\partial F}{\partial x}\right]'_{T/\alpha/\rho_a} \right\}, \\ \{J\}'_{T'/\alpha'/\rho'_a} &= -\left\{ [A]\left[\frac{\partial F}{\partial x}\right]'_{T'/\alpha'/\rho'} \right\}, \end{aligned} \tag{17}$$

by $[\ \ ]'_{T/\alpha/\rho_a}$ the differentiations with respect to T, α, ρ_a are denoted.

Denote

$$\left[\frac{\partial E}{\partial t}(u,t,x)\right]_{u_i,t_k,x_n} = \big[\dot{E}^k_{in}\big], \quad [J(u,t,x)]_{u_i,t_k,x_n} = \big[J^k_{in}\big], \tag{18}$$

$$J_i(t^k, x_{n+1/2}) = -\frac{1}{2}\left[\big(A^k_n + A^k_{n+1}\big)\frac{F^k_{n+1} - F^k_n}{\Delta x} \right]_i.$$

Taking into account Eqs. (14), (15) and the boundary conditions

$$\begin{aligned} &\frac{1}{8}\left(3\frac{\Delta E^k_0}{\Delta t} + \frac{\Delta E^k_1}{\Delta t}\right) + \frac{1}{\Delta x}\big(J^k_{1/2} - Q_0\big) = 0, \\ &\frac{1}{8}\left(3\frac{\Delta E^k_N}{\Delta t} + \frac{\Delta E^k_{N-1}}{\Delta t}\right) + \frac{1}{\Delta x}\big(Q_1 - J^k_{N-1/2}\big) = 0, \end{aligned} \tag{19}$$

we arrive at the iterative scheme of linear equations. If the i^{th} iteration of the solution u^k_{in} is known, then using Lagrange's formula $E^k_{i+1,n} = E^k_{in} + E^k_{1in}(u^k_{i,n} - u^k_{in})$, we obtain

$$I^k_{i+1,n+1/2} = I^k_{i,n+1/2} + I^k_{1in+1/2}\big(u^k_{i+1,n+1} - u^k_{in+1}\big) + I^k_{2i,n+1/2}\big(u^k_{i+1,n} - u^k_{in}\big).$$

Here E^k_{in}, $I^k_{i,n+1/2}$ are values of the vectors $\boldsymbol{E}_i$, $\boldsymbol{I}_i$ at the points (n,k), $(n+1/2,k)$.

To verify the result, we apply a slightly modified method of linearization, which is less time-consuming for the difference scheme. We proceed from Eqs. (11), (12), (16)–(19), where $Q_0 = (q_{h\,0}, q_{m\,0}, q_{a\,0})$ are the flows of enthalpy, moisture, and air through the surface $x = 0$. Then $q_{m\,0} = 0$, $q_{a\,0} = 0$, $q_{h\,0} = q_1 - \rho_p c_p h \Delta T^k_0/\Delta t$, N is the number of nodes on x; $Q_1 = (q_{h\,1}, q_{m\,1}, q_{a\,1})$ are flows through the surface $x = l$:

$$\begin{aligned} q_{h1} &= \frac{V}{S}\frac{\Delta}{\Delta t}\left[(\rho_{aN}c_{va} + \rho_{vN}c_{vv})T_N + r_0\rho^k_{vN}\right] + \frac{Q_e}{S(1+X_N)}\big[(X_N c_{pa} + c_{pv})T_N + r_0\big];\\ q_{m1} &= \frac{V}{S}\frac{\Delta}{\Delta t}\rho_{vN} + \frac{Q_e}{S(1+X_N)};\\ q_{a1} &= \frac{V}{S}\frac{\Delta}{\Delta t}\rho_{aN} + \frac{Q_e X_N}{S(1+X_N)}. \end{aligned} \tag{20}$$

Based on Lagrange's formula, we present

$$E^k_{i+1n} = E^k_{in} + E^k_{1in}\big(u^k_{i+1,n} - u^k_{i,n}\big), \tag{21}$$

$$J^k_{i+1,n+1/2} = J^k_{in+1/2} + J^k_{1in+1/2}\big(u^k_{i+1,n+1} - u^k_{i,n+1}\big) + J^k_{2in+1/2}\big(u^k_{i+1,n} - u^k_{i,n}\big),$$

where E^k_{in}, $J^k_{in+1/2}$ the values of the vectors $\boldsymbol{E}_i$, $\boldsymbol{J}_i$ at the points (n,k), $(n+1/2,k)$, respectively. E^k_{1in}, $J^k_{1in+1/2}$, $J^k_{2in+1/2}$ are matrices formed as follows:

$$E^k_{1in} = \left[\frac{\partial E^k_n}{\partial T^k_n}, \frac{\partial E^k_n}{\partial \alpha^k_n}, \frac{\partial E^k_n}{\partial \rho^k_{a\,n}}\right]_i, \quad J^k_{1in+1/2} = \left[\frac{\partial J^k_{n+1/2}}{\partial T^k_{n+1}}, \frac{\partial J^k_{n+1/2}}{\partial \alpha^k_{n+1}}, \frac{\partial J^k_{n+1/2}}{\partial \rho^k_{a\,n+1}}\right]_i,$$

$$J^k_{2in+1/2} = \left[\frac{\partial J^k_{n+1/2}}{\partial T^k_n}, \frac{\partial J^k_{n+1/2}}{\partial \alpha^k_n}, \frac{\partial J^k_{n+1/2}}{\partial \rho^k_{a\,n}}\right]_i. \tag{22}$$

By analogy

$$J^k_{i+1,n-1/2} = J^k_{in-1/2} + J^k_{1in-1/2}\big(u^k_{i+1,n-1} - u^k_{i,n-1}\big) + J^k_{2in-1/2}\big(u^k_{i+1,n} - u^k_{i,n}\big),$$

where $J^k_{in-1/2} = \big[J^k_{in-1/2}\big]$, $J^k_{in+1/2} = \big[J^k_{in+1/2}\big]$,

$$J^k_{1in-1/2} = \left[\frac{\partial J^k_{n-1/2}}{\partial T^k_{n-1}}, \frac{\partial J^k_{n-1/2}}{\partial \alpha^k_{n-1}}, \frac{\partial J^k_{n-1/2}}{\partial \rho^k_{a\,n-1}}\right]_i, \quad J^k_{2in-1/2} = \left[\frac{\partial J^k_{n-1/2}}{\partial T^k_n}, \frac{\partial J^k_{n-1/2}}{\partial \alpha^k_n}, \frac{\partial J^k_{n-1/2}}{\partial \rho^k_{a\,n}}\right]_i,$$

where

$$\frac{\partial J^k_{n+1/2}}{\partial T^k_{n+1}/\alpha^k_{n+1}/\rho^k_{a\,n+1}} = -\frac{1}{2}\left\{\frac{\partial A\left(u^k_{n+1}\right)}{\partial T^k_{n+1}/\alpha^k_{n+1}/\rho^k_{a\,n+1}}\left[\frac{F^k_{n+1} - F^k_n}{\Delta x}\right] - \frac{\left[A^k_n + A^k_{n+1}\right]}{\Delta x}\frac{\partial F^k_n}{\partial T^k_{n+1}/\alpha^k_{n+1}/\rho^k_{a\,n+1}}\right\},$$

$$\frac{\partial J^k_{n+1/2}}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}} = -\frac{1}{2}\left\{\frac{\partial A\left(u^k_n\right)}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}}\left[\frac{F^k_{n+1} - F^k_n}{\Delta x}\right] - \frac{\left[A^k_n + A^k_{n+1}\right]}{\Delta x}\frac{\partial F^k_n}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}}\right\},$$

$$\frac{\partial J^k_{n-1/2}}{\partial T^k_{n-1}/\alpha^k_{n-1}/\rho^k_{a\,n-1}} = -\frac{1}{2}\left\{\frac{\partial A\left(u^k_{n-1}\right)}{\partial T^k_{n-1}/\alpha^k_{n-1}/\rho^k_{a\,n-1}}\left[\frac{F^k_{n+1} - F^k_n}{\Delta x}\right] - \frac{\left[A^k_n + A^k_{n-1}\right]}{\Delta x}\frac{\partial F^k_n}{\partial T^k_{n-1}/\alpha^k_{n-1}/\rho^k_{a\,n-1}}\right\},$$

$$\frac{\partial J^k_{n-1/2}}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}} = -\frac{1}{2}\left\{\frac{\partial A\left(u^k_n\right)}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}}\left[\frac{F^k_n - F^k_{n-1}}{\Delta x}\right] - \frac{\left[A^k_n + A^k_{n-1}\right]}{\Delta x}\frac{\partial F^k_n}{\partial T^k_n/\alpha^k_n/\rho^k_{a\,n}}\right\}.$$

The variables $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$ are presented in the form:

$$[Q_0]^k_{i+1,0} = [Q_0]^k_{i,0} + [Q^1_0]^k_{i,0}\big(u^k_{i+1,0} - u^k_{i,0}\big) + [Q^2_0]^k_{i,0}\big(u^{k-1}_{i+1,0} - u^{k-1}_{i,0}\big),$$

$$[Q_0]^k_{i+1,N} = [Q_0]^k_{i,N} + [Q^1_1]^k_{i,N}\big(u^k_{i+1,N} - u^k_{i,N}\big) + [Q^2_1]^k_{i,N}\big(u^{k-1}_{i+1,N} - u^{k-1}_{i,N}\big), \tag{23}$$

where

$$[Q_0^1]_{i,0}^k = \left[\frac{\partial [Q_0]_{i,0}^k}{\partial T_0^k}, 0, 0\right]_i, \quad [Q_0^2]_{i,0}^k = \left[\frac{\partial [Q_0]_{i,0}^k}{\partial T_0^{k-1}}, 0, 0\right]_i,$$

$$[Q_1^1]_{i,N}^k = \left[\frac{\partial [Q_1]_{i,N}^k}{\partial T_N^k}, \frac{\partial [Q_1]_{i,N}^k}{\partial \alpha_N^k}, \frac{\partial [Q_1]_{i,N}^k}{\partial \rho_{a\,N}^k}\right]_i, \quad [Q_1^2]_{i,N}^k = \left[\frac{\partial [Q_1]_{i,N}^k}{\partial T_N^{k-1}}, \frac{\partial [Q_1]_{i,N}^k}{\partial \alpha_N^{k-1}}, \frac{\partial [Q_1]_{i,N}^k}{\partial \rho_{a\,N}^{k-1}}\right]_i.$$

Taking into account these ratios, Eqs. (16)–(19) are written as follows:

$$\left[\frac{3}{8\Delta t}E_{1i0}^k + \frac{1}{\Delta x}\left(J_{2i\frac{1}{2}}^k - (Q_0^1)_{i0}^k\right)\right](u_{i+10}^k - u_{i0}^k) - \left[\frac{3}{8\Delta t}E_{1i0}^{k-1} + \frac{1}{\Delta x}\left((Q_0^2)_{i0}^k\right)\right](u_{i+10}^{k-1} - u_{i0}^{k-1})$$
$$+ \left[\frac{1}{8\Delta t}E_{1i1}^k + \frac{1}{\Delta x}J_{1i\frac{1}{2}}^k\right](u_{i+11}^k - u_{i1}^k) - \frac{1}{8\Delta t}E_{1i1}^{k-1}(u_{i+11}^{k-1} - u_{i1}^{k-1})$$
$$= -\frac{1}{8\Delta t}\left[3\big(E_{10}^k - E_{10}^{k-1}\big) + \big(E_{11}^k - E_{11}^{k-1}\big)\right] - \frac{1}{\Delta x}\left(J_{i\frac{1}{2}}^k - (Q_0)_{i0}^k\right);$$

$$\left(\frac{1}{8\Delta t}E_{1in-1}^k - \frac{1}{\Delta x}J_{1in-\frac{1}{2}}^k\right)(u_{i+1n-1}^k - u_{in-1}^k) - \frac{1}{8\Delta t}E_{1in-1}^{k-1}(u_{i+1n-1}^k - u_{in-1}^k)$$
$$+ \left[\frac{6}{8\Delta t}E_{1in}^k + \frac{1}{\Delta x}\left(J_{2in+\frac{1}{2}}^k - J_{2in-\frac{1}{2}}^k\right)\right](u_{i+1n}^k - u_{in}^k) - \frac{6}{8\Delta t}E_{1in}^{k-1}(u_{i+1n}^{k-1} - u_{in-1}^{k-1})$$
$$+ \left[\frac{1}{8\Delta t}E_{1in+1}^k + \frac{1}{\Delta x}J_{1in+\frac{1}{2}}^k\right](u_{i+1n+1}^k - u_{in+1}^k) - \frac{1}{8\Delta t}E_{1in+1}^{k-1}(u_{i+1n+1}^{k-1} - u_{in+1}^{k-1})$$
$$= -\frac{1}{8\Delta t}\left[6\big(E_{in}^k - E_{in}^{k-1}\big) + \big(E_{in-1}^k - E_{in-1}^{k-1}\big) + \big(E_{in+1}^k - E_{in+1}^{k-1}\big)\right] - \frac{1}{\Delta x}\left(J_{in+\frac{1}{2}}^k - J_{in-\frac{1}{2}}^k\right); \quad (24)$$

$$\left[\frac{3}{8\Delta t}E_{1iN}^k - \frac{1}{\Delta x}\left(J_{2iN-\frac{1}{2}}^k - (Q_1^1)_{iN}^k\right)\right](u_{i+1N}^k - u_{iN}^k) - \left[\frac{3}{8\Delta t}E_{1iN}^{k-1} + \frac{1}{\Delta x}\left((Q_1^1)_{iN}^k\right)\right](u_{i+1N}^{k-1} - u_{iN}^{k-1})$$
$$+ \left[\frac{1}{8\Delta t}E_{1iN-1}^k - \frac{1}{\Delta x}J_{1iN-\frac{1}{2}}^k\right](u_{i+1N-1}^k - u_{iN-1}^k) - \frac{1}{8\Delta t}E_{1iN-1}^{k-1}(u_{i+1N-1}^{k-1} - u_{iN-1}^{k-1})$$
$$= -\frac{1}{8\Delta t}\left[3\left(E_{1N}^k - E_{1N}^{k-1}\right) + \left(E_{1N-1}^k - E_{1N-1}^{k-1}\right)\right] - \frac{1}{\Delta x}\left(J_{iN-\frac{1}{2}}^k + (Q_1)_{iN}^k\right),$$

where $[Q_0]_{i0}^k = \begin{bmatrix} q_e - \rho_p c_p h \frac{T_0^k - T_0^{k-1}}{\Delta t} \\ 0 \\ 0 \end{bmatrix}_i$,

$$\boldsymbol{Q}_1 = [b_i], \quad i = \overline{1,3},$$

$$b_1 = \frac{V}{S}\left\{\left(\frac{\partial \rho_a^0}{\partial t}C_{va} + \frac{\partial \rho_v^0}{\partial T}\frac{\partial T}{\partial t}C_{vv}\right)T + \left(\rho_a^0 C_{va} + \rho_v^0 C_{vv}\right)\frac{\partial T}{\partial t} + r_0\frac{\partial \rho_v^0}{\partial T}\frac{\partial T}{\partial t}\right\}$$
$$+ \frac{Q_e}{S}\frac{\left(\rho_a^0 C_{va} + \rho_v^0 C_{vv}\right)T + r_0\rho_v^0}{(\rho_a^0 + \rho_v^0)};$$

$$b_2 = \frac{1}{S}\frac{Q_e\rho_v^0}{\rho_v^0 + \rho_a^0} + \frac{V}{S}\frac{\partial \rho_v^0}{\partial T}\frac{\partial T}{\partial t}; \quad b_3 = \frac{1}{S}\frac{Q_e\rho_a^0}{\rho_v^0 + \rho_a^0} + \frac{V}{S}\frac{\partial \rho_a^0}{\partial t};$$

$$[Q_1]_{iN}^K = \begin{bmatrix} \frac{V}{S}\left\{C_{va}T_N^k\frac{\rho_{aN}^{0k} - \rho_{aN}^{0k-1}}{\Delta t} + \left[(C_{vv}T + r_0)\frac{\partial \rho_v^0}{\partial T} + (\rho_a^0 C_{va} + \rho_v^0 C_{vv})\right]_N^k \frac{T_N^k - T_N^{k-1}}{\Delta t}\right\} \\ + \frac{Q_e}{S}\frac{(\rho_a^0 C_{va} + \rho_v^0 C_{vv})T + r_0\rho_v^0}{(\rho_a^0 + \rho_v^0)}\bigg|_N^k \\ \frac{Q_e}{S}\frac{\rho_v^0}{(\rho_a^0 + \rho_v^0)}\bigg|_N^k + \frac{V}{S}\frac{\partial \rho_v^0}{\partial T}\bigg|_N^k \frac{T_N^k - T_N^{k-1}}{\Delta t} \\ \frac{Q_e}{S}\frac{\rho_v^0}{(\rho_a^0 + \rho_v^0)}\bigg|_N^k + \frac{V}{S}\frac{\rho_{a\,N}^{0\,k} - \rho_{a\,N}^{0\,k-1}}{\Delta t} \end{bmatrix}_i,$$

$$\frac{\partial Q_{1\,N}^{k}}{\partial T_N^k}=\begin{bmatrix} \frac{V}{S}\left\{C_{va}\frac{\rho_{aN}^{0k}-\rho_{aN}^{0k-1}}{\Delta t}+\left[(C_{vv}T+r_0)\frac{\partial\rho_v^0}{\partial T}+\left(\rho_a^0C_{va}+\rho_v^0C_{vv}\right)\right]_N^k\frac{1}{\Delta t}\right\}+ \\ +\frac{T_N^k-T_N^{k-1}}{\Delta t}\left[\frac{\partial^2\rho_v^0}{\partial T^2}(C_{vv}T+r_0)+2C_{vv}\frac{\partial\rho_v^0}{\partial T}\right]_N^k+\frac{Q_e}{S}\left\{\left[\frac{(\rho_a^0C_{va}+\rho_v^0C_{vv})T+r_0\rho_v^0}{(\rho_a^0+\rho_v^0)^2}+\right.\right. \\ \left.\left.+\frac{(C_{pv}T+r_0)}{(\rho_a^0+\rho_v^0)}\right]\frac{\partial\rho_v^0}{\partial T}\Big|_N^k+\frac{(\rho_a^0C_{pa}+\rho_v^0C_{pv})}{(\rho_a^0+\rho_v^0)}\Big|_N^k\right\} \\ \frac{Q_e}{S}\frac{\rho_v^0}{(\rho_a^0+\rho_v^0)^2}\frac{\partial\rho_v^0}{\partial T}\Big|_N^k+\frac{V}{S}\frac{\partial\rho_v^0}{\partial T}\Big|_N^k\frac{1}{\Delta t}+\frac{T_N^k-T_N^{k-1}}{\Delta t}\left(\frac{\partial^2\rho_v^0}{\partial T^2}\right)_N^k \\ -\frac{Q_e}{S}\frac{\rho_a^0}{(\rho_a^0+\rho_v^0)^2}\frac{\partial\rho_v^0}{\partial T}\Big|_N^k \end{bmatrix}_i,$$

$$\frac{\partial Q_{1\,N}^{k}}{\partial \alpha_N^k}=0,\quad \frac{\partial Q_{1\,N}^{k}}{\partial \alpha_N^{k-1}}=0,$$

$$\frac{\partial Q_{1\,N}^{k}}{\partial T_N^{k-1}}=\begin{bmatrix} -\frac{V}{S}\left[(C_{vv}T+r_0)\frac{\partial\rho_v^0}{\partial T}+(\rho_a^0C_{va}+\rho_v^0C_{vv})\right]_N^k\frac{1}{\Delta t} \\ -\frac{V}{S}\frac{\partial\rho_v^0}{\partial T}\Big|_N^k\frac{1}{\Delta t} \\ 0 \end{bmatrix};$$

$$\frac{\partial Q_{1\,N}^{k}}{\partial \rho_{a\,N}^k}=\begin{bmatrix} \frac{V}{S}\left\{C_{va}T_N^k\frac{1}{\Delta t}+C_{va}\frac{T_N^k-T_N^{k-1}}{\Delta t}\right\}+\frac{Q_e}{S}\left[\frac{(C_{pa}T)}{(\rho_a^0+\rho_v^0)}-\frac{(\rho_a^0C_{pa}+\rho_v^0C_{pv})T+r_0\rho_v^0}{(\rho_a^0+\rho_v^0)^2}\right]_N^k \\ -\frac{Q_e}{S}\frac{\rho_v^0}{(\rho_a^0+\rho_v^0)^2}\Big|_N^k \\ -\frac{Q_e}{S}\frac{\rho_v^0}{(\rho_a^0+\rho_v^0)^2}\Big|_N^k+\frac{V}{S}\frac{1}{\Delta t} \end{bmatrix};$$

$$\frac{\partial Q_{1\,N}^{k}}{\partial \rho_{a\,N}^{k-1}}=\begin{bmatrix} -\frac{V}{S}C_{va}T_N^k\frac{1}{\Delta t} \\ 0 \\ \frac{1}{\Delta t} \end{bmatrix};\quad \frac{\partial Q_{0\,0}^{k}}{\partial T_0^k}=\begin{bmatrix} -\rho_pC_ph\frac{1}{\Delta t}\Big|_0^k \\ 0 \\ 0 \end{bmatrix}_i;\quad \frac{\partial Q_{0\,0}^{k}}{\partial T_0^{k-1}}=\begin{bmatrix} \rho_pC_ph\frac{1}{\Delta t}\Big|_0^{k-1} \\ 0 \\ 0 \end{bmatrix};$$

$$\frac{\partial Q_{0\,0}^{k}}{\partial \alpha_0^k}=0;\quad \frac{\partial Q_{0\,0}^{k}}{\partial \alpha_0^{k-1}}=0;\quad \frac{\partial Q_{0\,0}^{k}}{\partial \rho_{a\,0}^k}=0;\quad \frac{\partial Q_{0\,0}^{k}}{\partial \rho_{a\,0}^{k-1}}=0;$$

$$\frac{\partial E_N^k}{\partial \alpha_N^k}=\begin{bmatrix} \left\{T\Pi\left[C_L\rho_L-(\rho_a^0C_{va}+\rho_v^0C_{vv})\right]-r_0\Pi\rho_v^0\right\}_N^k \\ -\Pi(\rho_L^0-\rho_v^0)\big|_N^k \\ -\Pi\rho_a^0\big|_N^k \end{bmatrix},$$

$$\frac{\partial E_N^k}{\partial T_N^k}=\begin{bmatrix} \left\{\rho_0C_0+\Pi\left[C_L\rho_L\alpha+(1-\alpha)(\rho_a^0C_{va}+\rho_v^0C_{vv})\right]_N^k\right\}+\Pi(1-\alpha)(TC_{vv}+r_0)\frac{\partial\rho_v^0}{\partial T}\Big|_N^k \\ \Pi(1-\alpha)\frac{\partial\rho_v^0}{\partial T}\Big|_N^k \\ 0 \end{bmatrix},$$

$$\frac{\partial E_N^k}{\partial \rho_{a\,N}^k}=\begin{bmatrix} T\Pi(1-\alpha)C_{va}\big|_N^k\left\{\rho_0C_0+\Pi\left[C_L\rho_L\alpha+(1-\alpha)(\rho_a^0C_{va}+\rho_v^0C_{vv})\right]_N^k\right\} \\ +\Pi(1-\alpha)(TC_{vv}+r_0)\frac{\partial\rho_v^0}{\partial T}\Big|_N^k \\ 0 \\ \Pi(1-\alpha)\big|_N^k \end{bmatrix},$$

$$\boldsymbol{F}=\begin{bmatrix} T \\ \alpha \\ \frac{\rho_v^0}{\rho_v^0+\rho_a^0} \\ \frac{\rho_v^0}{\rho_v^0+\rho_a^0} \end{bmatrix}.$$

$$\frac{\partial A_N^k}{\partial T_N^k}=\begin{bmatrix} 0 & 0 & \frac{\tilde K}{\mu_{eff}}(1-\alpha)\Big[\left(r_0\rho_v^0+TC_{pa}\rho_a^0+TC_{pv}\rho_v^0\right) & \Pi D_{ef}(1-\alpha)(\rho_v^0+\rho_a^0)(C_{pv}-C_{pa}) \\ & & +C_{pa}\rho_a^0+C_{pv}\rho_v^0+(TC_{pv}+r_0)\frac{\partial\rho_v^0}{\partial T}\Big]_N^k & +\frac{\partial\rho_v^0}{\partial T}(r_0-C_{pa}T+C_{pv}T)\Big|_N^k \\ \frac{\partial D_L}{\partial T}\rho_0\delta & \frac{\partial D_L}{\partial T}\rho_L & \frac{\tilde K}{\mu_{eff}}(1-\alpha)\frac{\partial\rho_v^0}{\partial T}\Big|_N^k & \Pi D_{ef}(1-\alpha)\frac{\partial\rho_v^0}{\partial T}\Big|_N^k \\ 0 & 0 & 0 & \Pi D_{ef}\frac{\tilde K}{\mu_{eff}}(1-\alpha)\rho_v^0\Big|_N^k- \end{bmatrix};$$

$$\frac{\partial A_N^k}{\partial \alpha_N^k}=\begin{bmatrix} \frac{\partial\lambda}{\partial\alpha} & 0 & -\frac{\tilde K}{\mu_{eff}}\Big[(r_0\rho_v^0+TC_{pa}\rho_a^0+TC_{pv}\rho_v^0) & -\Pi D_{ef}(\rho_v^0+\rho_a^0)(r_0-C_{pa}T+C_{pv}T) \\ & & +C_{pa}\rho_a^0+C_{pv}\rho_v^0++(TC_{pv}+r_0)\frac{\partial\rho_v^0}{\partial T}\Big]_N^k & +\frac{\partial\rho_v^0}{\partial T}(r_0-C_{pa}T+C_{pv}T)\Big|_N^k \\ \Big(\frac{\partial D_L}{\partial\alpha}\ \rho_0\delta+ & \frac{\partial D_L}{\partial\alpha}\Pi\rho_L & \frac{\tilde K\rho_L}{\mu_L}-\frac{\tilde K\rho_v^0}{\mu_{ef}}\Big|_N^k & -\Pi D_{ef}(\rho_v^0+\rho_a^0)\big|_N^k \\ +a_L\rho_0\frac{\partial\delta}{\partial\alpha}\ \frac{\partial a_L}{\partial\alpha}\Big)\Big|_N^k & & & \\ 0 & 0 & -\frac{\tilde K\rho_a^0}{\mu_{ef}}\Big|_N^k & \Pi D_{ef}(\rho_v^0+\rho_a^0)\big|_N^k \end{bmatrix};$$

$$\frac{\partial A_N^k}{\partial \rho_N^k}=\begin{bmatrix} 0 & 0 & -\frac{\tilde K}{\mu_{eff}}(1-\alpha)Tc_{pa} & -\Pi D_{ef}(1-\alpha)(r_0-C_{pa}T+C_{pv}T) \\ 0 & 0 & 0 & \Pi D_{ef}(1-\alpha) \\ 0 & 0 & \frac{\tilde K}{\mu_{ef}}(1-\alpha)\Big|_N^k & -\Pi D_{ef}(1-\alpha)\big|_N^k \end{bmatrix};$$

$$\frac{\partial F_N^k}{\partial T_N^k}=\begin{bmatrix} 1 \\ 0 \\ \left(\frac{\rho_v^0}{M_v}+\frac{\rho_a^0}{M_a}\right)R+\frac{\partial\rho_v^0}{\partial T}\frac{RT}{M_v} \\ \frac{\rho_a^0}{(\rho_v^0+\rho_a^0)^2}\frac{\partial\rho_v^0}{\partial T} \end{bmatrix}_N^k; \quad \frac{\partial F_N^k}{\partial \alpha_N^k}=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_N^k, \quad \frac{\partial F_N^k}{\partial \rho_a^0{}_N^k}F=\begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_a}RT_N^k \\ -\frac{\rho_v^0}{(\rho_v^0+\rho_a^0)^2} \end{bmatrix}_N^k.$$

The transfer coefficients are taken from the Luikov's work [19]. This model describes mass transfer processes under moderate heat loads.

The capillary-porous material for which

$$\lambda=\begin{cases} \lambda_0+(\lambda_L-\lambda_0)\sin\frac{\pi\alpha}{2\alpha_\lambda}, & \alpha\leqslant\alpha_\lambda, \\ \lambda_L, & \alpha>\alpha_\lambda. \end{cases}$$

Here $\lambda_L=0.06\,\mathrm{W/(m\ K)}$, $\alpha_\lambda=0.1$;

$$\frac{\partial\lambda}{\partial\alpha}=\begin{cases} \frac{\pi}{2\alpha_\lambda}(\lambda_L-\lambda_0)\cos\frac{\pi\alpha}{2\alpha_\lambda}, & \alpha<\alpha_\lambda, \\ 0, & \alpha>\alpha_\lambda; \end{cases}$$

$$D_L(T,\alpha)=\begin{cases} \left[D_{L0}+(D_{LL}-D_{L0})\sin\frac{\pi\alpha}{2\alpha_a}\right](T/273)^{20}, & \alpha<\alpha_D, \\ D_{LL}(T/273)^{20}, & \alpha>\alpha_D; \end{cases}$$

$$\frac{\partial D_L(T,\alpha)}{\partial T}=\begin{cases} \left[D_{L0}+(D_{LL}-D_{L0})\sin\frac{\pi\alpha}{2\alpha_a}\right](20T^{19}/273^{20}), & \alpha<\alpha_D, \\ D_{LL}(20T^{19}/273^{20}), & \alpha<\alpha_D; \end{cases}$$

$$\frac{\partial D_L(T,\alpha)}{\partial\alpha}=\begin{cases} \frac{\pi}{2\alpha_a}(D_{LL}-D_{L0})\cos\frac{\pi\alpha}{2\alpha_a}(T/273)^{20}, & \alpha<\alpha_D, \\ 0, & \alpha<\alpha_D; \end{cases}$$

$$\delta(\alpha)=\delta_0\left[1-4(\alpha-0.5)^2\right], \quad \frac{\partial\delta(\alpha)}{\partial\alpha}=\delta_0(4-8\alpha);$$

$$\frac{\partial\rho_v^0}{\partial T}=133\frac{M_v}{RT}\exp\left(18.681-\frac{4105}{T-35}\right)\left[\frac{4105}{(T-35)^2}-\frac{1}{T}\right],$$

$$\frac{\partial^2\rho_v^0}{\partial T^2}=133\frac{M_v}{RT}\exp\left(18.681-\frac{4105}{T-35}\right)\left[\left(\frac{4105}{(T-35)^2}-\frac{1}{T}\right)^2-\left(\frac{2\cdot 4105}{(T-35)^3}-\frac{1}{T^2}\right)\right];$$

$$P_c=\left(\frac{\rho_v^0}{M_v}+\frac{\rho_a^0}{M_a}\right)TR, \quad \frac{\partial P_c}{\partial\rho_v^0}=\frac{RT}{M_v}, \quad \frac{\partial P_c}{\partial\rho_a^0}=\frac{RT}{M_a}, \quad \frac{\partial P_c}{\partial T}=\left(\frac{\rho_v^0}{M_v}+\frac{\rho_a^0}{M_a}\right)R.$$

4. Results and discussion

Calculations are made for an aluminum plate and a capillary-porous material of different porosity (cork tree). The flow of the continuous phase is assumed to be slow. Inertial terms are neglected. Transfer coefficients are considered to be known functions of saturation and temperature, $T_0 = 290\,\mathrm{K}$, $h = 2 \cdot 10^{-3}\,\mathrm{m}$, $D_{LL} = 1.5 D_{L0}$, $\alpha_0 = 0.2$, $P_0 = 1.01325 \cdot 10^5\,\mathrm{Pa}$, $R = 8.31\,\mathrm{J/Kmol}$, $M_v = 1.8 \cdot 10^{-3}\,\mathrm{kg/mol}$, $M_a = 2.9 \cdot 10^{-3}\,\mathrm{kg/mol}$, $C_{pa} = 1.006 \cdot 10^3\,\mathrm{J/(kg{\cdot}K)}$, $C_{pv} = 1.103 \cdot 10^3\,\mathrm{J/(kg{\cdot}K)}$, $C_{va} = 718\,\mathrm{J/(kg{\cdot}K)}$, $C_{vv} = 862\,\mathrm{J/(kg{\cdot}K)}$, $\alpha_L = 9.5 \cdot 10^{-1}$, $r_0^* = 2.3 \cdot 10^6\,\mathrm{J/kg}$, $P_e = 10\,\mathrm{Pa}$, $l = 5 \cdot 10^{-2}\,\mathrm{m}$, $L = 3 \cdot 10^{-2}\,\mathrm{m}$, $\Pi = 9 \cdot 10^{-1}$, $C_L = 4.190 \cdot 10^3\,\mathrm{J/(kg{\cdot}K)}$, $C_0 = 10^3\,\mathrm{J/(kg{\cdot}K)}$, $\tilde{K} = 10^{-14}\,\mathrm{m}^2$, $\mu_L = 5 \cdot 10^{-4}\,\mathrm{kg/(m{\cdot}s)}$, $\mu_{ef} = \mu_g = 10^{-5}\,\mathrm{kg/(m{\cdot}s)}$, $D_{ef} = 5 \cdot 10^{-5}\,\mathrm{m^2/s}$, $\lambda_0 = 6 \cdot 10^{-2}\,\mathrm{W/(m{\cdot}K)}$, $\lambda_L = 6 \cdot 10^{-1}\,\mathrm{W/(m{\cdot}K)}$, $\alpha_g = 8.5 \cdot 10^{-1}$, $\alpha_\lambda = \alpha_a = 10^{-1}$, $\rho_0 = 6 \cdot 10^{-2}\,\mathrm{kg/m^3}$, $\rho_L = 10^3\,\mathrm{kg/m^3}$, $\delta_0 = 10^{-3}\,\mathrm{1/K}$, $D_{L0} = 10^{-3}\,\mathrm{m^2/s}$, $T_c = 327\,\mathrm{K}$, $s/S = 10^{-4}$, $V/S = 3 \cdot 10^{-2}\,\mathrm{m}$.

A comparison of the accuracy of the calculation results is given in Table 1.

Table 1. Temperature and moisture calculation results (division along $x = 4$, $t = 5$).

The first method of linearization		The second method of linearization	
Temperature	Moisture	Temperature	Moisture
297.17	0.59	297.17	0.59
297.03	0.58	297.03	0.58
290.01	0.57	290.01	0.57
267.71	0.72	267.70	0.72
304.49	0.57	304.50	0.57
301.83	0.55	301.83	0.55
284.67	0.56	284.66	0.56
259.74	0.81	259.72	0.81
311.52	0.54	311.52	0.54
304.07	0.53	304.08	0.53
279.45	0.57	279.44	0.57
257.16	0.87	257.14	0.87
317.80	0.50	317.80	0.50
304.88	0.50	304.88	0.50
275.91	0.59	275.90	0.59
256.85	0.92	256.85	0.92
323.06	0.46	323.06	0.46
305.28	0.49	305.28	0.49
274.05	0.60	274.04	0.60
255.93	0.95	255.91	0.95

As an example, porous materials with the porosity $\Pi = 0.4$, 0.6, and 0.8 heated by heat flows $q = 3 \cdot 10^3$, $5 \cdot 10^3$, 10^4 are considered and the influence of various parameters on drying processes is investigated. The results of the calculations are shown in Figures 2–7.

The solutions of the problem are obtained by finite-difference and iterative methods, and the comparison of the results of these solutions is used to study their accuracy.

Calculations show that depending on the amount of heat flow, porosity, and initial saturation of the capillary-porous material, evaporation proceeds in different ways. Concerning the dependence of temperature on porosity, under the action of the flow $q = 10^4$, $5 \cdot 10^3$, $3 \cdot 10^3$ and $\Pi = 0.4$, 0.6 during $5 \cdot 10^2$ s, the temperature is a monotonically increasing function; but at $q = 10^4$ and $\Pi = 0.8$, this dependence is no longer monotonous either inside the material or on its surfaces. With these parameters already at $1.5 \cdot 10^2$ s, moisture of a certain mass is released from the material, while the temperature first drops slightly and then increases over time more slowly than in a material with the same characteristics but with lower porosity. At the same time, the less the porosity, the greater the growth gradient.

At the same porosity and the amount of heat flow, at the beginning of the evaporation process, the temperature increases faster with a lower initial moisture content. Figures 2, 6, and 7 demonstrate

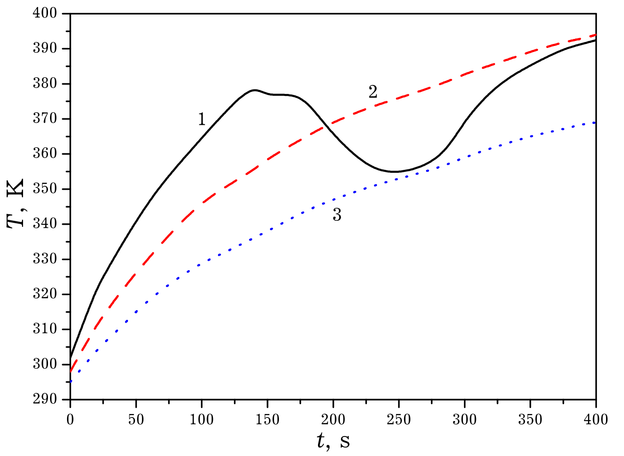

Fig. 2. Temperature variations in time for $\Pi = 0.9$ and $\alpha_0 = 0.8$. The curves 1, 2, 3 correspond to the $q = 10^4$, $5 \cdot 10^3$, $3 \cdot 10^3$, respectively.

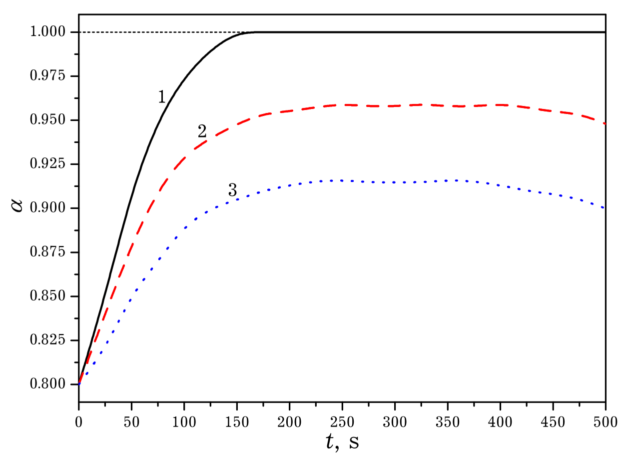

Fig. 3. Change in volumetric saturation in time on the outer surface for $\alpha_0 = 0.8$. The curves 1, 2, 3 correspond to the $q = 3 \cdot 10^3$, $5 \cdot 10^3$, 10^4, respectively.

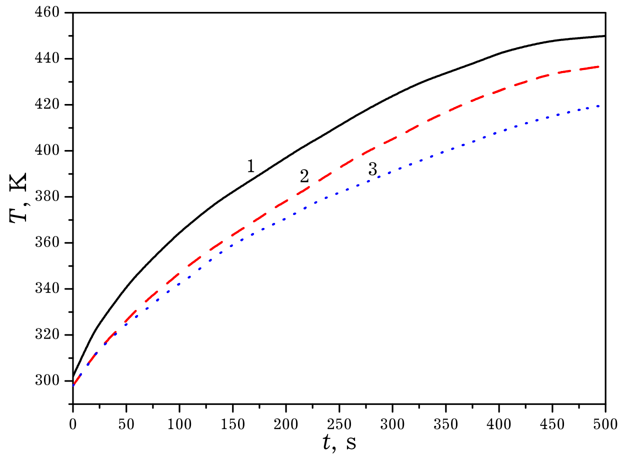

Fig. 4. Temperature variations in time under the action of the flow $q = 5 \cdot 10^3$. The curves 1, 2, 3 correspond to the porosity $\Pi = 0.4$, 0.6, 0.8, respectively.

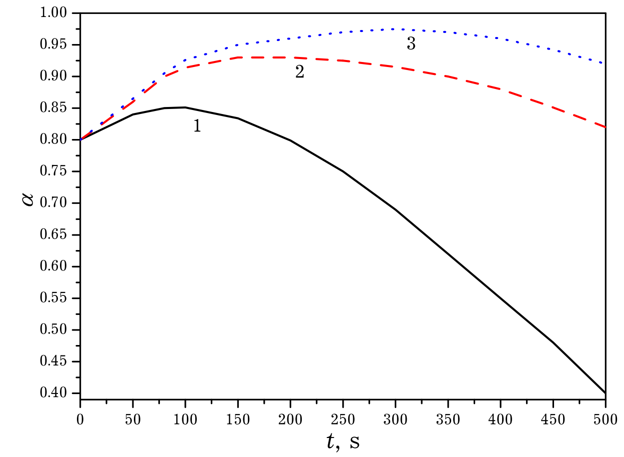

Fig. 5. Change in volumetric saturation in time on the outer surface for $q = 5 \cdot 10^3$, $\alpha_0 = 0.8$. The curves 1, 2, 3 correspond to the porosity $\Pi = 0.4$, 0.6, 0.8, respectively.

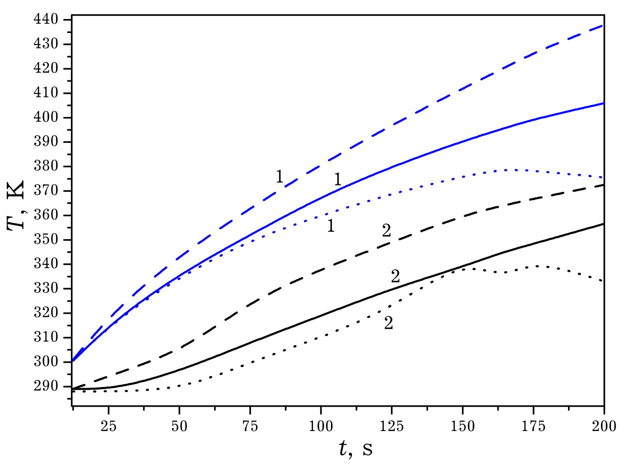

Fig. 6. Temperature variations in time on the heating surface (curves 1); external surface (curves 2) for $q = 10^4$, $\alpha_0 = 0.8$ for different values of porosity (dashed curves for $\Pi = 0.4$; solid curves for $\Pi = 0.6$, dotted curves for $\Pi = 0.8$).

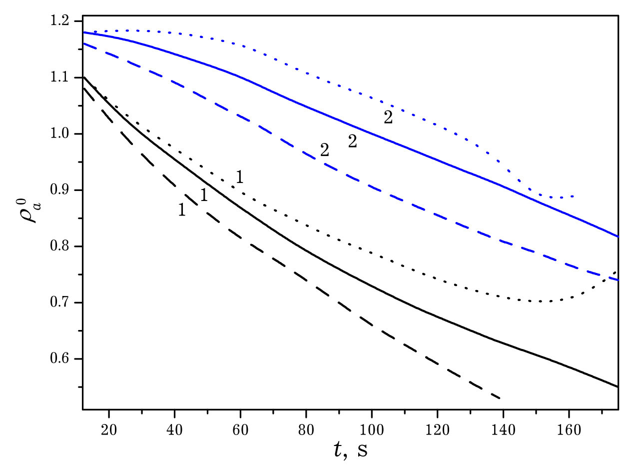

Fig. 7. Change in air density in time on the heating surface (curves 1); external surface (curves 2) for $q = 10^4$, $\alpha_0 = 0.8$ for different values of porosity (dashed curves for $\Pi = 0.4$; solid curves for $\Pi = 0.6$, dotted curves for $\Pi = 0.8$).

the change in temperature over time for different heat flows and different porosities; Figures 4, 5 show a change in time of relative saturation for the same flows and porosities in the process of an irregular regime of conductive drying. Studies of the influence of the initial saturation for the heat flow intensity of $q = 5 \cdot 10^3\,\mathrm{W/m^2}$ and the porosity of the material $\Pi = 0.8$ have shown that the greater the initial saturation, the faster it increases at the beginning of drying and the slower it decreases over time. At the initial saturation $\alpha = 0.8$ at the beginning of the process, the saturation increases faster from the side of the plate heating at a porosity of $\alpha = 0.6$ and more slowly for $\Pi = 0.4$. Graphs of temperature variations in time show that the greater the porosity, the slower the temperature increases on the surface from the heating side and from the cavity side. At the same time, with porosity $\Pi = 0.8$, the temperature variation for the flow rate of $10^4\,\mathrm{W/m^2}$ has an oscillatory character.

5. Conclusions

In the article, a system of essentially non-linear differential equations of heat and mass transfer for the dense packing of capillary-porous materials based on the approaches of the theory of a mixture of porous and dense packing of dispersed materials of multicomponent three-phase media is obtained. The change in the characteristics of the phases making up the body, depending on the temperature and the initial relative saturation, affects the behavior of both the temperature and the saturation of the porous body during contact drying of the material. These characteristics are especially affected in the first stage of drying when the influence of the initial conditions is important. Therefore, the phenomena occurring at the stage of heating, with high initial moisture content, are considered.

Calculations have shown that, depending on the magnitude of the heat flux, porosity, and initial saturation of the capillary-porous material, evaporation proceeds differently. The temperature (dependent on porosity) under the action of the flow $q = 3 \cdot 10^3$, $5 \cdot 10^3$, $10^4\,\mathrm{W/m^2}$ with the porosity $\Pi = 0.4$, 0.6 during 500 s is a monotonically increasing function of time, but for $q = 10^4\,\mathrm{W/m^2}$ and $\Pi = 0.8$, this dependence is no longer monotonous either inside or on the surfaces of the material. With a heat flux $q = 10^4\,\mathrm{W/m^2}$ and the porosity $\Pi = 0.8$, already at the 150^{th} second of drying, moisture of a certain mass is released from the material (condensation caused by oncoming warm and cold flows), while the temperature first decreases slightly and then increases with time slower than in a material with the same characteristics but with less porosity. In this case, the lower the porosity, the greater the gradient of temperature rise. This property is used in problems of thermal protection of materials. With the same porosity and heat flow at the beginning of the evaporation process, the temperature increases faster with a lower initial moisture content of the material.

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Числове моделювання процесів тепломасопереносу в капілярно-пористому тілі при контактному осушенні

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Розглянуто проблему кондуктивного (контактного) сушіння капіляро-пористого тіла в пароповітряному (газовому) середовищі передачею теплоти до матеріалу при контакті його з нагрітими поверхнями матеріалу. Отримано систему суттєво нелінійних диференціальних рівнянь тепломасоперенесення для опису такого процесу. Для розв'язування сформульованої задачі тепломасоперенесення (без врахування деформативності) застосовано методику розв'язування нелінійних крайових задач у вигляді ітераційного процесу, на кожному кроці якого розв'язується лінійна крайова задача. Проведено перевірку результатів методу двома способами. Вони добре узгоджуються. Проведено чисельний експеримент для матеріалів трьох видів пористості. Результати представлено графічно та таблично. Виведено закономірності контактного сушіння капіляро-пористих матеріалів в пароповітряному середовищі.

Ключові слова: *контактне сушіння; капілярно-пористий матеріал; система нелінійних диференціальних рівнянь; ітераційний процес; лінійна крайова задача.*