Vol. 5, No. 1, 2023

Petro Kosobutskyy[1], Volodymyr Karkulovskyy[2]

[1]Computer-Aided Design Department, Lviv Polytechnic National University,
St. Bandery str.,12, Lviv, Ukraine, , E-mail: petro.s.kosobutskyy@lpnu.ua, ORCID 0000-0003-4319-7395
[2]Computer-Aided Design Department, Lviv Polytechnic National University,
St. Bandery str., 12, Lviv, Ukraine, E-mail:volodymyr.i.karkulovskyy@lpnu.ua, ORCID0009-0007-0787-5482

RECURRENCE AND STRUCTURING OF SEQUENCES OF TRANSFORMATIONS 3N + 1 AS ARGUMENTS FOR CONFIRMATION OF THE COLLATZ HYPOTHESIS

Received: June 10, 2023 / Revised: August 28, 2023 / Accepted: September 28, 2023

https://doi.org/

Abstract. It is shown that infinites of the subsequence of odd numbers is not a counterargument of the violation of the Collatz hypothesis, but a universal characteristic of transformations of natural numbers by the 3n + 1 algorithm. A recurrent relationship is established between the parameters of the sequence of Collatz transformations of an arbitrary pair of natural numbers n and 2n.

Keywords: Collatz conjecture, recurrent sequence, transformation 3n + 1, natural numbers.

Introduction

In 1937, L. Kollatz [1] formulated the well-known problem of transforming the set of natural numbers N by functions

$$a_n = \begin{cases} \frac{1}{2}a_{n-1}\ if\ a_{n-1}\ is\ even \quad (a) \\ 3a_{n-1}+1\ if\ a_{n-1}\ is\ odd \quad (b) \end{cases}. \tag{1}$$

According to the algorithm (1), an arbitrary even number is divided by 2, and an odd number is multiplied by 3 and 1 is added to the result. The well-known Kollatz hypothesis states that the process (1) always ends with a periodic sequence with a single value of the lower limit of the oscillation amplitude:

$$... \to 4 \to 2 \to 1. \tag{2}$$

In 1976, R.Terras [3,4] introduced the data compression algorithm (1) and twice proved that the result of multiple integration of the function is always smaller than the initial value of the number , and in 2018 T. Tao [5] significantly strengthened this result by proving that . However, the validity of Kollatz's conjecture is established for a finite number of natural numbers, so it is not proven, and in the future it remains only an assertion. Kollatz's conjecture is also known as (Kakutani's problem), (the Syracuse algorithm), (Hasse's algorithm), (Thwaites conjecture), (Ulam's problem). The chronology of solving the Kollatz problem can be found in [6].

Usually, Kollatz transformations (1) are carried out by the method of counting numbers, and the limitation of computing resources forms the upper limit of its value [7], which so far did not allow us to make a confident conclusion about a cycle of stopping calculations other than (2). Currently, this problem is undergoing multifaceted research [8]. However, from time to time there are publications with counterarguments about the violation of Kollatz's hypothesis, as recently, for example [9]. Professional reviews of works like [9] are given in [10].

Problem Statement

This paper systematizes the regularities of transformations of odd numbers, including those accepted in [9] as a counterargument for the violation of Kollatz's hypothesis. In addition, the recurrent relationship between the number of iterations (the length of the Kollatz sequence (KS)) of the transformations of a pair of numbers and from the start (1) to the complete stop (2) is substantiated.

Let's formulate the lemmas and the theorem:

Lemma 1.On the set of equal natural numbers, the numbers are evenly distributed.

Lemma 2. Doubles and singles of even and odd numbers are similar to each other.

Lemma 3.. With a non-stop cycle of Collatz transformations

$$a_n = \begin{cases} \frac{1}{2}a_{n-1} \; if \; a_{n-1} \; is \; even \\ 3a_{n-1}+1 \; if \; a_{n-1} \; is \; odd \end{cases}$$ *pairs and natural numbers in the set , does not exist.*

Let us substantiate Lemma 3. To do this, consider the sequence on a binary basis

$$\{K_{\theta=1}\}: \quad 1\cdot\{1=2^0, 2=2^1, 4=2^2, 8=2^3, 16=2^4, 32=2^5, 64=2^6, 128=2^8, ...\}, \tag{3}$$

- is the stem (the main graph) at the points of which the equality holds

$$3q+1=2^n-1 \Rightarrow \frac{2^n-1}{3} = \text{int}\,eger, \tag{4}$$

and nodes are formed, the coordinates of which form a sequence

$$\theta=1, \{m_k\}: \quad \{0, 1, 5, 21, 85, 341, ...\}, \quad (a) \quad \Rightarrow \quad m_k = \frac{1\cdot 4^k-1}{3}, \quad (b) \quad k=0,1,2,3,4,,...., \tag{5}$$

which is known as A002450 [2].

The sequence (3) is formed in accordance with the laws of Newton's binomial unfolding

$$(1+1)^n, \; n=0,1,2,3,..., \tag{6}$$

and on its basis Pascal's triangle (Fig. 1). The binary sequence (3) is recurrent and is a solution of the equation

$$(1+1)^n, \; n=0,1,2,3,... \tag{7}$$

The inset shows the graph multiplication rule (9), and the inset shows the recurrent rule for adding the numbers of neighboring nodes, highlighted by a dotted block, from which other graphs with an arbitrary positive value and initial conditions are subsequently multiplied according to the scenario depicted in Fig. 1

$K_0=1, K_1=2$. Really, $2^{n+2} = \varepsilon\cdot 2^{n+1} - 2(\varepsilon-2)\cdot 2^n \Rightarrow 4 = \varepsilon\cdot 2 - 2(\varepsilon-2) = 2\varepsilon - 2\varepsilon + 4 = 4$.

Nodes (5) are branching points of new linear one-dimensional graphs (branches of the so-called Kollatz tree). Thus, the coordinates of nodes on the lowest graph form a sequence A072197 [2]:

$$\{m_{5,k}\}: \quad \{3, 13, 53, 213, ...\}, \quad (a) \quad \Rightarrow \quad m_{5,k} = \frac{5\cdot 2^k-1}{3}, \quad (b) \quad k=1, 3, 5, 8, ..., \tag{8}$$

Therefore, side graphs branch off from the main graph (3). In Fig. 1, graphs with nodes, including (3), are marked as *TYPE 1*. Graphs *TYPE 1* are generated by points whose coordinates are 5, 85, 341, 5461, etc., which are not divisible by three. Graphs and *TYPE 2* without nodes, generated by points whose coordinates are 21, 1365, etc., which are divisible by three.

$$K_\theta = \theta\cdot K_{\varpi=1}, \tag{9}$$

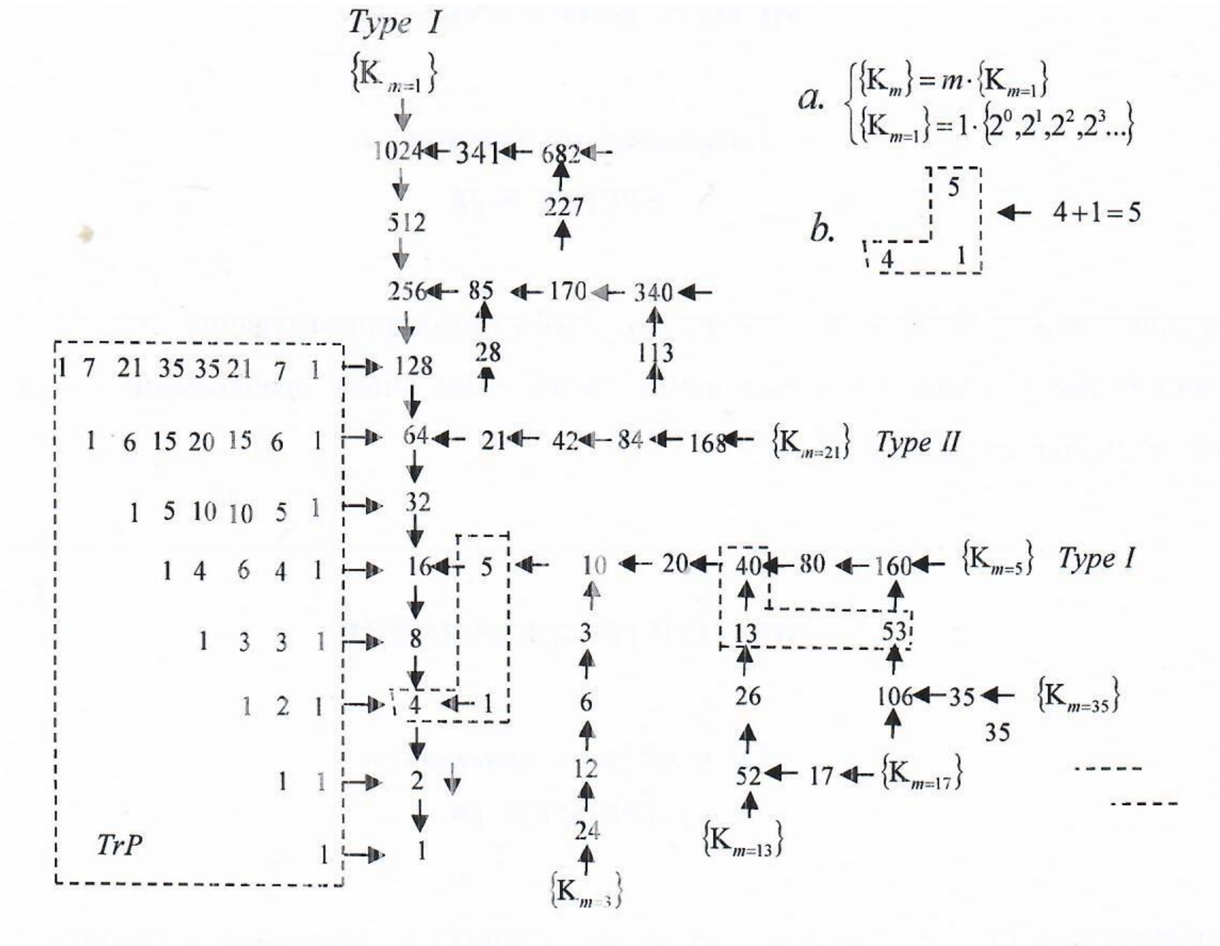

Fig. 1. Illustration of Pascal's triangle on a tree of Collatz graphs and branches with nodes (Type I) and without nodes (Type II).

This division is conditional, and is related to the structure of the odd numbers themselves, which in Fig. 2 are highlighted with a darker background. In KS, they play the role of peculiar limits that unilaterally limit the subsequence of odd numbers. Thus, to the left of these limits, only graphs of subsequences of doubled numbers are formed. If the given odd numbers are prime.

$$5,7,11,13,17,19,23,25,29,31,\ldots, \tag{10}$$

arrows ← From below, the sequences are limited to odd numbers for which the trajectory of the transformations in the direction of a complete stop is known.

If they are not factored, graphs of subsequences of doubled numbers (the upper part above the odd numbers in Fig. 2) are formed with nodes. If the odd numbers are added

$$3,9,15,21,27,33,39,45\ ,\ldots\ . \tag{11}$$

When factored, there are no nodes on them. For example, in PC numbers 26208: 26208→13104→6552→3276→1638→819→2458→1229→3688→1844→922→461→1384→692→ →346→173→520→260→130→65→196→98→49→148→74→37→112→56→28→14→7→22→11→3 4→17→→52→26→13→40→20→10→5→16→8→4→2→1 subsequence of odd numbers

$$\begin{gathered}\to 819\to\ldots\to 1229\to\ldots\to 461\to\ldots\to 173\to\ldots\to 65\to 49\to\\ \ldots\to 37\to\ldots\to 7\to\ldots\to 17\to\ldots\to 13\to\ldots\to 5\to\ \ldots\to 1,\end{gathered} \tag{12}$$

bounded on the left by an odd complex number 819=32·91. Therefore, to the left of the number 819a subsequence of doubled numbers is formed without knots.

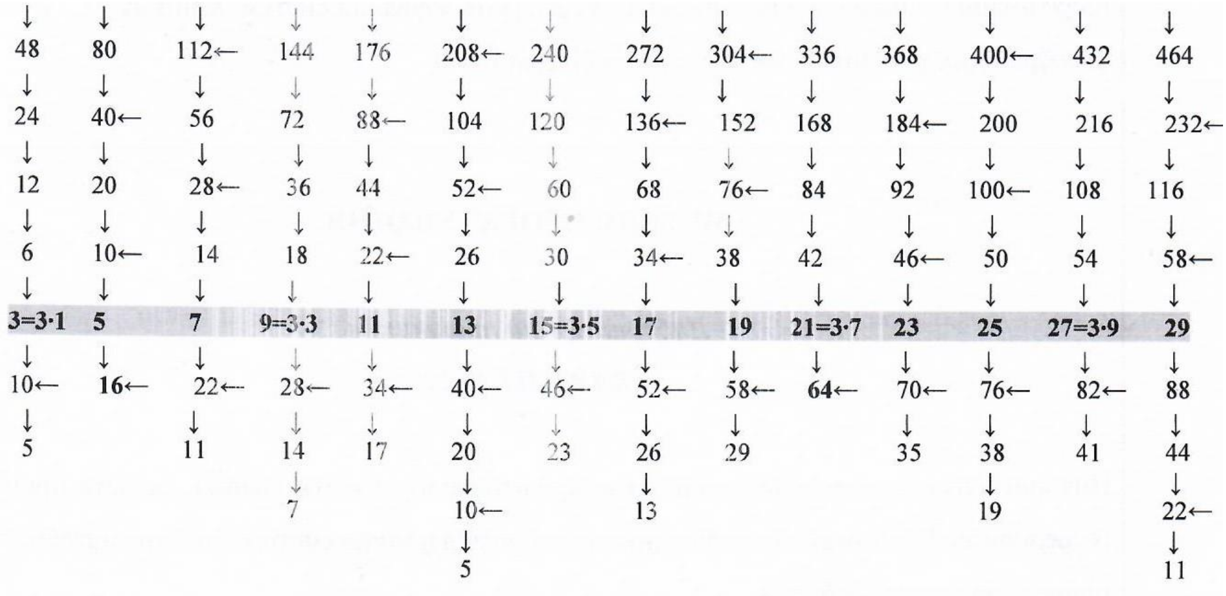

Fig. 2. Illustration of KS of simple (10) and complex (11) odd numbers.

Nodes are shown with arrows ←. From below, the sequences are limited to odd numbers for which the trajectory of the transformations in the direction of a complete stop is known.

The number *s* the first composite number generated by the Collatz algorithm in the subsequence of odd numbers of type (12). In [9], a similar regularity for the composite number 27=33 was used as a counterargument for the violation of Kollatz's hypothesis. In fact, as it follows from Fig. 2, one-sided restriction of subsequences of odd numbers takes place for all prime and composite odd numbers, so it is a universal characteristic of KS formation.

In KS, in the direction of direct transformation (1), every odd number is surrounded by two even numbers a_{s-1} та a_{s+1}. Therefore, the KS can be represented in the form of interconnected blocks $... \to a_{s-1} \to a_s \to a_{s+1} \to ...$. To the left of the limiting odd number a_s, the inequality holds $a_{s-1} \rangle a_{s+1}$, but rather to the right $a_{s-1} \langle a_{s+1}$, де s - direct conversion number (1) from the start time for the given number. In the block $... \to a_{s+1} \to a_s \to a_{s-1} \to ...$ odd number can be eliminated as: $... \to a_{s-1} \to (a_{s+1} - a_{s-1} - 1) \to a_{s+1} \to ...$

3105168421
63105168421
1263105168421
241263105168421
48241263105168421
9648241263105168421
1929648241263105168421

Fig. 3. PC illustration for the first seven doubled three numbers

Results and discussion

With one-sided restriction of the subsequence of odd composite numbers, the subsequence of even numbers at $n \to \infty$ increases. We will show that this growth of the subsequence of even numbers is also not a counterargument for the violation of Kollatz's hypothesis. To do this, we binary structure a number of natural numbers according to the doubling algorithm

$$0(2\cdot n),\ 2^0(2\cdot n),\ 2^1(2\cdot n),\ 2^2(2\cdot n),\ 2^3(2\cdot n),\ 2^4(2\cdot n),\ 2^5(2\cdot n),\ 2^6(2\cdot n),\ 2^7(2\cdot n),\ .2^8(2\cdot n)\ ..., \quad (13)$$

and calculate the Collatz conjucture (or Hailstone or) [11] KS for the first doubled three numbers by aligning them along the right edge, as shown in Fig. 3. Taking as a quantitative characteristic of transformation (1) the number of iterations, which is equivalent to the time of a complete stop of the process or the length of the trajectory of transformations (1) from Fig. 3, we make sure that and are connected to each other by a recurrence relation

$$N_{2n} = N_n + n\,(a) \quad \Rightarrow \quad N_{2n} - N_n = n\,(b). \quad (14)$$

Thus, formula (14) removes the problem of going to infinity of the Kollatz sequence for even numbers [9], since it is determined by the transition $n \to \infty$, while N_n at the same time, remains finished.

Binary structuring of the set $n \in \infty$ according to the algorithm (13) does not change the similarity of subsets n and $2n$ (lemma 2). Since lemma 2 does not require proof, then KS with quantitative characteristics N_n i N_{2n} similar numbers n i $2n$, are also similar to each other for the entire set of numbers $n \in \infty$. Therefore, the recurrent formulas (14) are true for the entire set $n \in \infty$.

By lemma 1, all normal numbers n equal to each other. Therefore, according to (14b), KS are equal to each other N_n i N_{2n}, and their growth $N_{2n} - N_n$ proportional to the number itself n. On the whole set $n \in \infty$ numbers n are uniform and equal to each other, therefore, according to (14), KS are similar to each other on the entire set $n \in \infty$, and have a single cycle (2) stopping the Kollatz calculations. *The lemma is justified*.

From the successive application of formula (14a) for many pairs of doubled numbers

$$N_{s\cdot n} = (n + 2n + 4n + \cdots + sn) + N_n, s = 2,4,8,12, ..., \quad (15)$$

the recurrent formula follows in its general form

$$N_{s\cdot n} - N_n = (2^s - 1)\cdot n, \quad (16)$$

when $(2^s - 1)$ - these are known Mersenne numbers [14]. A formula similar to (15) was obtained in [12] by dividing the set $n \in \infty$ on subsets of even and odd numbers. However, their similarity in terms of the degree of PC fusion was not proven in [12] [13]. According to the degree of fusion, a conclusion was made about the equivalence of PC for the numbers 3 and 13. [9]. But, as you can see from Fig. 2, KS numbers 3 and 13 are fundamentally different.

In conclusion, we note the following. For the direct transformation (1), after the node there is a merging of calculation trajectories, as, for example, for the numbers 224 and 74 after the node with coordinate 112 in the third column in Fig. 2 in the direction of the complete stop of the calculation process. However, in the reverse direction after starting from point 1, for the algorithm (Definition 4.1) [9] this node and others are undefined. This means that the task of modeling the trajectories of Kollatz calculations when passing a node in the reverse direction requires additional information regarding the choice of the direction of the movement trajectory

Conclusions

The limitation of the subsequence of odd numbers is not a counterargument for the violation of Kollatz's hypothesis, but a universal characteristic of transformations of natural numbers according to the 3n+1 algorithm. The lengths of the Kollatz sequences of transformations of an arbitrary pair of natural numbers n and 2n are recursively connected to each other by the values of the single number n. Thus, the established regularities of PC formation confirm the fulfillment of Collatz's hypothesis. The work has a practical application in the field of automated design of complex objects and systems to optimize the design strategy, taking into account the specifics of the design situation.

References

1.L. Collatz On the motivation and origin of the (3n + 1) – Problem, J. Qufu Normal University, Natural Science Edition,1986, 12(3), 9–11.

2. The On-line encyclopedia of integer sequences. The OEIS Foundation is supported by donations from users

of the OEIS and by a grant from the Simons Foundation, https://oeis.org/A002450

3. R. Terras. A stopping time problem on the positive integers. Acta Arith. 1976, 30, 241–252. https://doi.org/10.4064/aa-30-3-241-252

4.Terras, R. On the Existence of a Density. Acta Arith. 1979, 35, 101–102. https://doi.org/10.4064/aa-35-1-101-102

5. T. Tao. Almost all orbits of the Collatz map attain almost bounded values, 2020.

6.A. Rahn, E. Sultanow, M. Henkel, S. Ghosh, I. Aberkane. An Algorithm for Linearizing the Collatz Convergence.Mathematics 2021, 9(16), 1898; https://doi.org/10.3390/math9161898

7.Collatz conjucture. Collatz best results, https://www.dcode.fr/collatz-conjecture

8. J. Kleinnijenhuis, A. Kleinnijenhuis, M. Aydogan. The Collatz tree as a Hilbert hotel: a proof of the 3x + 1 conjecture/arXiv:2008.13643v3 [math.GM] 17 Jan 2021

9. M. Ahmed Maya. Two different scenarios when the Collatz Conjecture fails, arXiv:2001.04976v1 [math.GM]

10. L. Green. LOG book, http://lesliegreen.byethost3.com/publications.html

11.Collatz Conjucture, https://www.dcode.fr/collatz-conjecture.

12. C.Cadogan C. A Solution to the 3x + 1 Problem Charles Cadogan.юCaribb. J. Math. Comput. Sci. — 2006. — 13, p. 1-11

13. И.Рысцов. Замечание по поводу доказательства гипотезы Коллатца. Украинский математический конгресс – 2009. (UKRAINIAN MATHEMATICAL CONGRESS − 2009 (Dedicated to the Centennial of Nikolai N. Bogoliubov), Kyiv, Institute of Mathematics of NASU, August 27−29, 2009). https://www.imath.kiev.ua/~congress2009/Abstracts/Rystcov.pdf

14. P.Catarino, H.Campos, P.Vasco. On the Mersenne sequence. Annales Mathematicae et Informaticae, vol.46 (2016), 37-53

Петро Кособуцький[1], Володимир Каркульовський[2]

[1]Кафедра систем автоматизованого проектування, Національний університет Львівська політехніка, вул. Степана Бандеи 12, Львів, Україна, E-mail: petro.s.kosobutskyy@lpnu.ua, ORCID 0000-0003-4319-7395

[2]Кафедра систем автоматизованого проектування, Національний університет Львівська політехніка, вул. Степана Бандеи 12, Львів, Україна, E-mail:volodymyr.i.karkulovskyy@lpnu.ua, ORCID0009-0007-0787-5482

ЗАКОНОМІРНОСТІ ФОРМУВАННЯ ПОСЛІДОВНОСТЕЙ 3N + 1 ПЕРЕТВОРЕНЬ ЯК АРГУМЕНТ ПІДТВЕРДЖЕННЯ ГІПОТЕЗИ КОЛЛАТЦА

Отримано: липень 10, 2023 / Переглянуто: серпень 28, 2023 / Прийнято: вересень 28, 2023

Анотація. Показано, що необмеженість підпослідовності непарних чисел не контраргумент порушення гіпотези Коллатца, а універсальна характеристика перетворень натуральних чисел за алгоритмом 3n+1. Встановлений рекурентний зв’язок між параметрами послідовності Коллатца перетворень довільної пари натуральних чисел n і 2n **.**

Ключові слова: гіпотеза Коллатца, послідовність, перетворення 3n+1, натуральні числа.