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The behaviours of convergents in θ-expansions: computational insights based on θ-expansions algorithm using the Maple software

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Continued fractions arise naturally in long division and the theory of the approximation to real numbers by rational numbers. This research considered the implementation on the convergent of θ-expansions of real numbers of $x \in (0, \theta)$ with $0 < \theta < 1$. The convergent of θ-expansions are also called as θ-convergent of continued fraction expansions. This study aimed to establish the properties for a family of θ-convergent in θ-expansions. The idea of discovering the behaviours of θ-convergent evolved from the concept of regular continued fraction (RCF) expansion and sequences involved in θ-expansions. The θ-expansions algorithm was used to compute the values of θ-convergent with the help of Maple software. Consequently, it proved to be an efficient method for fast computer implementation. The growth rate of θ-convergent was investigated to highlight the performance of θ-convergent. The analysis on θ-convergent revealed the convergent that gives a better approximation yielding to fewer convergence errors. This whole paper thoroughly derived the behaviours of θ-convergent, which measure how well a number x is approximated by its convergents for almost all irrational numbers.

Keywords: *θ-convergent; θ-expansions; θ-expansions algorithm; continued fraction; convergence errors.*

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1. Introduction

In many years, continued fractions had evolved along with the curiosity in solving many problems including expanded irrational square roots, computed continued fraction of differential equations by Lagrange's method [1], computed continued fraction of π [2], convergence theory [3] and others. Thus, variants of continued fractions were created in the quest to explore and to establish more results related to continued fractions.

A continued fraction $\langle\langle\{a_n\},\{b_n\}\rangle,\{x_n\}\rangle$ is often denoted by the expression as follows:

$$x_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots + \frac{a_n}{b_n}}}.$$

The numbers a_n and b_n are called the nth partial numerator and denominator of the continued fraction, respectively. Sometimes, they are simply called the elements; x_n is called the nth approximant. If $\{a_n\}$ and $\{b_n\}$ are infinite sequences, then $\langle\langle\{a_n\},\{b_n\}\rangle,\{x_n\}\rangle$ is called an infinite (or non-terminating) continued fraction. It is called a finite (or terminating) continued fraction if $\{a_n\}$ and $\{b_n\}$ have only a finite number of terms $a_1, a_2, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$.

The origin of continued fractions is traditionally placed at the time of the creation of Euclid's algorithm [4]. The Euclidean algorithm involves the division of the larger integer by the smaller, the smaller by the remainder, the first remainder by the second remainder, and so on until the exact division is obtained whence the greatest common divisor (GCD) is the exact divisor.

An Indian Mathematician, Aryabhata, used a continued fraction to solve a linear indeterminate equation [4], which refers to the equation that cannot be directly solved from the given information. For instance,

$$ax + by = c, \quad x^2 - Py^2 = 1.$$

Rather than generalizing this method, the author used the continued fractions in specific examples. Continued fraction became a field in its right through the work of John Wallis [5]. The theory concerning continued fractions was significantly developed, especially that concerning the convergents. The convergent of a continued fraction is presented as follows [6].

Definition 1. *The continued fraction made from $[a_0, a_1, a_2, \ldots, a_n]$ by cutting off the expansion after the kth partial denominator a_k is called the kth convergent of the given continued fraction and denoted by C_k; in symbols,*

$$C_k = [a_0, a_1, \ldots, a_k], \quad 0 \leqslant k \leqslant n.$$

Let the zeroth convergent C_0 be equal to the number a_0.

Wallis [7] established many of the basic properties of convergents in his book entitled *Arithmetica Infinitorum.* Next, Bosma and Kraaikamp [6] derived the distribution of the sequence $\theta_n(x)_{n\geqslant1}$, which measures how well a number x is approximated by its convergents for almost all irrational numbers. Subsequently, Elsner and Komatsu [8] studied the leaping convergents $\frac{p_{3n+1}}{q_{3n+1}}$ for the continued fraction of $e = \lfloor 2; (1, 2k, 1)\rfloor_{k=1}^{\infty}$ and leaping convergents $\frac{p_{3n}}{q_{3n}}$ for the continued fraction of $e^{\frac{1}{s}} = \lfloor 1; (s(2k-1)-1, 1, 1)\rfloor_{k=1}^{\infty}$ where $s \geqslant 2$. They obtained some arithmetic properties and extended such results for further general continued fractions. As one of the applications, they showed a new recurrence formula for leaping convergents of Apery's continued fraction of $\zeta(3)$.

Since the beginning of the 20th century, continued fractions have made their appearances in other fields. For instance, much research has examined the connection between continued fractions and other fields relative to the chaos theory. Besides that, continued fractions are being utilized within computer algorithms for computing rational approximations to real numbers as well as solving indeterminate equations. Numerous studies, including solving Pell's equation using continued fractions [9], applying continued fractions in the field of cryptography [10, 11], and continued fraction expansions that contribute to the Gauss map [12], invariant measure [13], Lebesgue measure [13], and ergodicity in dynamical systems [14], show the development of this subject.

In previous works, many researchers explored the development of various types of continued fractions such as regular continued fraction (RCF) [15], generalized continued fraction (GCF) [16], nearest integer continued fraction (NICF) [17], semi-regular continued fraction (SRCF) [18], optimal continued fraction (OCF) [6], Engel continued fraction (ECF) [13], and θ-expansions [19]. θ-expansions have contributed to solving many problems, especially in the ergodic system [20–24]. The idea of θ-expansions has been motivated and implemented by RCF expansions which contributed to many applications. One of the most important uses is called the Gauss–Kuzmin theorem in the metrical theory of θ-expansions. Previous research has solved a Gauss–Kuzmin problem for θ-expansions [21, 22]. They applied the theory of random systems with complete connections extensively studied by Iosifescu and Grigorescu [25] and the method of Rockett and Szusz [26] to solve the Gauss–Kuzmin problem.

Recently, the view of θ-expansions has been expanded through the studies made by Muhammad and Kamarulhaili [27]. Based on the samples in that research, it was clearly shown that $0 < \theta < 1$ affects the behavior of θ-expansions and tends to make it different from the RCF expansions. From that paper, they have listed the similarities and differences for both θ and RCF expansions. Hence, in this paper, our concern is to critically examine the behaviors of convergents in θ-expansions and observe the performance of $0 < \theta < 1$ towards the growth rate of convergents.

2. Mathematical preliminaries

Chakraborty and Rao [19] solved the problems in random continued fraction expansions related to θ-expansions. The problem includes the symbolic dynamics of the map $T_\theta\colon [0,\theta] \to [0,\theta]$ and the existence of absolutely continuous invariant probability. The value of θ is fixed within $0 < \theta < 1$ and $x \in (0,\theta)$, where x is referred to the irrational number, $x \in \mathbb{Q}^c \in \mathbb{R}$. The θ-expansion is defined as follows [27].

Definition 2. *For any $x \in (0,\theta)$ and $0 < \theta < 1$, let x be the θ-continued fraction expansion of x, and its expansion is as follows:*

$$x = a_0\theta + \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \cfrac{1}{\ldots + \cfrac{1}{a_n\theta + \cfrac{1}{R_n}}}}}}, \tag{1}$$

where $R_n = [a_{n+1}\theta, a_{n+2}\theta, a_{n+3}\theta, \ldots]$ and a_n's are non-negative integers. The expansion in equation (1) is also called the θ-expansion of x with $a_0\theta = 0$ and $a_n \in \mathbb{R}$ where $a_n > 0$. Such a_n's are called the θ-continued fraction digits of x with respect to this expansion.

Equation (1) can be written as $x = [a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}]$. If $x \in \mathbb{Q}$, then the θ-expansion is finite as x has a finite θ-expansion. Meanwhile, if $x \in \mathbb{Q}^c$, then the θ-expansion is infinite as x has an infinite θ-expansion. This statement means those real numbers expressible as finite θ-expansion are precisely the rational numbers, while those real numbers expressible as infinite θ-expansion are precisely the irrational numbers. So, all real numbers have representations as θ-expansion, where the rational numbers are characterized by having finite representations and the irrational numbers are characterized by having infinite representations. Hence, the expansion in equation (1) is an infinite θ-expansion or non-terminating expansion. While, the finite θ-expansion will terminate at the nth step and the expansion is as follows:

$$x = a_0\theta + \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{\ldots + \cfrac{1}{a_n\theta}}}}. \tag{2}$$

Then, equation (2) can be written as $x = [a_0\theta, a_1\theta, a_2\theta, \ldots, a_n\theta]$. Next, the convergent of θ-expansions is defined as follows [19].

Definition 3. *For any $x \in (0,\theta)$ and $0 < \theta < 1$,*

$$C_n = [a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta]$$

is defined as the n-th order of θ-convergent of continued fraction expansions.

Much of the labor in calculating the θ-convergents can be avoided by establishing formulas for their numerators and denominators. Hence, this leads to the following recurrence relations [27].

Theorem 1. *Let $x = [a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}]$, where $R_n = [a_{n+1}\theta, a_{n+2}\theta, a_{n+3}\theta, \ldots]$ be an infinite θ-expansion of continued fraction. Then, we have recurrence relations for $n \in \mathbb{N}$:*

$$\begin{gathered} p_0 = a_0\theta, \quad p_1 = a_0\theta a_1\theta + 1, \quad p_n = a_n\theta p_{n-1} + p_{n-2}; \\ q_0 = 1, \quad q_1 = a_1\theta, \quad q_n = a_n\theta q_{n-1} + q_{n-2}. \end{gathered} \tag{3}$$

Theorem 2. *Let $x = [a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}]$, where $R_n = [a_{n+1}\theta, a_{n+2}\theta, a_{n+3}\theta, \ldots]$ be an infinite θ-expansion of continued fraction. Define $p_0, p_1, p_2, \ldots, p_n$ and $q_0, q_1, q_2, \ldots, q_n$ by the recurrence relations of equation (3). Then, we have the n-th order of θ-convergent of continued fraction expansions based on those recurrence relations as*

$$C_n = \frac{p_n}{q_n}. \tag{4}$$

From the equations (1), (3), and (4), θ-expansion now has n terms, which is

$$x_n = \frac{p_n + \frac{1}{R_n} p_{n-1}}{q_n + \frac{1}{R_n} q_{n-1}}, \tag{5}$$

with $R_n = [a_{n+1}\theta, a_{n+2}\theta, a_{n+3}\theta, \ldots] = x_{n+1}$. Based on the recurrence relations in equation (3), we have the following property [27].

Lemma 1. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion of a continued fraction, where $R_n = x_{n+1}$ with $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Then,*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad \text{for} \quad n \geqslant 0. \tag{6}$$

Next, from Theorem 1, we obtain the following consequences as indicated in the following corollaries.

Corollary 1. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion of a continued fraction, where $R_n = x_{n+1}$. $C_n = [a_0\theta, a_1\theta, a_2\theta, \ldots, a_n\theta] = \frac{p_n}{q_n}$ is defined as θ-convergent, where $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Hence, we have*

$$C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_n q_{n-1}}, \quad n \geqslant 1. \tag{7}$$

Corollary 2. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion of a continued fraction, where $R_n = x_{n+1}$. $C_n = [a_0\theta, a_1\theta, a_2\theta, \ldots, a_n\theta] = \frac{p_n}{q_n}$ is defined as θ-convergent, where $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Hence, we have*

$$C_n - C_{n-2} = \frac{(-1)^n a_n\theta}{q_n q_{n-2}}, \quad n \geqslant 2. \tag{8}$$

Corollary 1 and Corollary 2 are important in the extension of the finite θ-expansions to the infinite θ-expansions, which are obtained by expanding the θ-convergent $C_n = \frac{p_n}{q_n}$.

3. Results and discussions

In this section, we will discuss the behaviours and examine the patterns of θ-convergent with their numerical computations.

3.1. Theoretical behaviours of θ-convergent

Corollary 1 and Corollary 2 give us important information that the θ-convergent, C_n changes as n increases. θ-convergent can be defined as the even θ-convergent or the odd θ-convergent. The n-th order of θ-convergent will verify whether the θ-convergent is even or odd. Even θ-convergent occurred when the value of n of n-th order is an even number such as C_2, C_4, C_6, $C_8, \ldots$. While, odd θ-convergent occurred when the value of n of n-th order is an odd number such as C_1, C_3, C_5, $C_7, \ldots$. The previous two corollaries lead to the following theorems, regarding the properties of the even θ-convergent and the odd θ-convergent.

Theorem 3. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion of a continued fraction where $R_n = x_{n+1}$. $C_n = [a_0\theta, a_1\theta, a_2\theta, \ldots, a_n\theta] = \frac{p_n}{q_n}$ is defined as θ-convergent where $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Subsequently, every value of even θ-convergent is less than every value of odd θ-convergent.*

Proof. By Corollary 1, we have

$$C_n - C_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}}. \tag{9}$$

Then, we substitute equation (6) into equation (9), we obtain

$$C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_n q_{n-1}}, \quad n \geqslant 1.$$

For the even case, if we set $n = 2$, then we obtain

$$C_2 - C_1 = \frac{(-1)^1}{q_2 q_1} < 0, \tag{10}$$

which satisfies q_n is always positive. Then, from equation (10), we obtain

$$C_2 < C_1. \tag{11}$$

Now, we set n as even with $n > 2$ using the same formula as in equation (7), and we observe

$$C_4 < C_3, \quad C_6 < C_5, \quad C_8 < C_7, \quad C_{10} < C_9, \quad \dots . \tag{12}$$

Next, for the odd case, using $n = 3$ and substituting into equation (7), we obtain

$$C_3 - C_2 = \frac{(-1)^2}{q_3 q_2} > 0, \tag{13}$$

which satisfies q_n is always positive. From equation (13), we have

$$C_3 > C_2. \tag{14}$$

Then, we set n as odd with $n > 3$, by Corollary 1, we will obtain the following inequalities:

$$C_5 > C_4, \quad C_7 > C_6, \quad C_9 > C_8, \quad C_{11} > C_{10}, \quad \dots . \tag{15}$$

Based on equations (11), (12), (14), (15), we found that every value of even θ-convergent is always less than every value of odd θ-convergent as desired. ■

Theorem 4. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \dots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion of a continued fraction where $R_n = x_{n+1}$. $C_n = [a_0\theta, a_1\theta, a_2\theta, \dots, a_n\theta] = \frac{p_n}{q_n}$ is defined as θ-convergent where $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Hence, we have*

1. *The even θ-convergent, C_0, C_2, C_4, ... forms a strictly increasing sequence of real numbers, such that*

$$C_0 < C_2 < C_4 < \dots . \tag{16}$$

2. *The odd θ-convergent, C_1, C_3, C_5, ... forms a strictly decreasing sequence of real numbers, such that*

$$C_1 > C_3 > C_5 > \dots . \tag{17}$$

Proof. By Corollary 2, we have

$$C_n - C_{n-2} = \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}.$$

By simplifying the above equation, we have

$$C_n - C_{n-2} = \frac{a_n\theta(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})}{q_n q_{n-2}}. \tag{18}$$

Next, when substituting equation (6) into equation (18), we obtain

$$C_n - C_{n-2} = \frac{a_n\theta(-1)^{n-2}}{q_n q_{n-2}} = \frac{(-1)^n a_n\theta}{q_n q_{n-2}}, \quad n \geqslant 2.$$

For the even case, if we set $n = 2$, we will have

$$C_2 - C_0 = \frac{(-1)^2 a_2\theta}{q_2 q_0} > 0, \tag{19}$$

where q_n is always positive. From equation (19), we will obtain

$$C_2 > C_0. \tag{20}$$

We set n as even with $n > 2$ step-by-step following the above fashion, we obtain the following inequalities:

$$C_0 < C_2 < C_4 < \dots . \tag{21}$$

From equation (21), we can see that the even θ-convergent forms a strictly increasing sequence of real numbers as desired. Next, for the odd case, if we set $n = 3$, we will have

$$C_3 - C_1 = \frac{(-1)^3 a_3 \theta}{q_3 q_1} < 0, \tag{22}$$

where q_n is always positive. From equation (22), we obtain

$$C_3 < C_1. \tag{23}$$

Then, we set n as odd with $n > 3$ using the same step as in equation (22) and we obtain the following inequalities:

$$\dots < C_5 < C_3 < C_1. \tag{24}$$

From equation (24), we found that the odd θ-convergent forms a strictly decreasing sequence of real numbers as desired. ■

We can regard an infinite θ-expansion, $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \dots, a_n\theta + \frac{1}{R_n}\right]$ as an analog of an infinite decimal expansion of a real number. Truncating an infinite decimal expansion of an irrational number produces a rational approximation of the number. The following property of θ-convergent allows us to determine how good the approximation of it on x_n.

Theorem 5. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \dots, a_n\theta + \frac{1}{R_n}\right]$ be an infinite θ-expansion where $R_n = x_{n+1}$ with $x \in (0, \theta)$ and fixed θ within the range of $0 < \theta < 1$. We set $x_n = \frac{p_n + \frac{1}{R_n} p_{n-1}}{q_n + \frac{1}{R_n} q_{n-1}}$ with the relations of p_n and q_n as $p_n = a_n\theta p_{n-1} + p_{n-2}$ and $q_n = a_n\theta q_{n-1} + q_{n-2}$. Hence, we have*

$$\left| x_n - \frac{p_n}{q_n} \right| < \left| \frac{\pm 1}{q_n q_{n+1}} \right|. \tag{25}$$

Proof. From equation (1), we have $x = [a_0\theta, a_1\theta, a_2\theta, a_3\theta, \dots, a_n\theta + \frac{1}{R_n}]$ where $R_n = x_{n+1}$:

$$x = a_0\theta + \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \dots + \cfrac{1}{a_n\theta + \frac{1}{R_n}}}}.$$

Considering this θ-expansion having n terms, we note that $x = x_n$. Then, applying the formula for the n-th θ-convergent. We are interested in the size of the difference between x_n and $\frac{p_n}{q_n}$, that is $x_n - \frac{p_n}{q_n}$, so from equation (5), we can write

$$x_n - \frac{p_n}{q_n} = \frac{R_n p_n + p_{n-1}}{R_n q_n + q_{n-1}} - \frac{p_n}{q_n}.$$

By simplifying the above equation, we have

$$x_n - \frac{p_n}{q_n} = \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n (R_n q_n + q_{n-1})}.$$

From Lemma 1, we have $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$, which means that $p_{n-1}q_n - p_n q_{n-1} = \pm 1$. Therefore,

$$\left| x_n - \frac{p_n}{q_n} \right| = \left| \frac{\pm 1}{q_n (R_n q_n + q_{n-1})} \right|.$$

Now, since $R_n > a_n\theta$ because $a_n\theta \in \mathbb{R}$ and $R_n \in \mathbb{R}$ is the same real number plus the rest of the θ-expansions. Now, note that since decreasing the size of the denominator results in making the fraction larger, then we have

$$\left| x_n - \frac{p_n}{q_n} \right| < \left| \frac{1}{q_n (a_{n+1}\theta q_n + q_{n-1})} \right|.$$

Since we have $a_{n+1}\theta q_n + q_{n-1} = q_{n+1}$, so now we have

$$\left| x_n - \frac{p_n}{q_n} \right| < \left| \frac{\pm 1}{q_n q_{n+1}} \right|$$

as desired. ■

From Theorem 5, we have the following property:

Corollary 3. *If $\frac{p_n}{q_n}$ is the nth θ-convergent to the irrational number x, then*

$$\left|x_n - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{26}$$

Proof. From Theorem 5, we have proved that

$$\left|x_n - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

Then, now we need to compare between $q_n q_{n+1}$ and q_n^2 where we have the following equations:

$$q_n q_{n+1} = q_n(q_{n+1}),$$
$$q_n^2 = q_n(q_n).$$

From the behavior of q_n, we have $q_n > q_{n-1}$, which gives us $q_{n+1} > q_n$ and the following inequality:

$$q_n q_{n+1} > q_n^2. \tag{27}$$

So, from equation (27), we obtain

$$\frac{1}{q_n^2} > \frac{1}{q_n q_{n+1}}. \tag{28}$$

Then, from equation (5), we obtain $x_n - \frac{p_n}{q_n}$ as follows:

$$\left|x_n - \frac{p_n}{q_n}\right| = \left|\frac{R_n p_n + p_{n-1}}{R_n q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-1}q_n - p_n q_{n-1}}{q_n(R_n q_n + q_{n-1})}\right|.$$

From Lemma 1, we have $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$. Therefore, we have

$$\left|x_n - \frac{p_n}{q_n}\right| = \left|\frac{1}{q_n(R_n q_n + q_{n-1})}\right|. \tag{29}$$

Now, we have the denominator of equation (28) and equation (29) as follows:

$$q_n^2 = q_n(q_n),$$
$$q_n(R_n q_n + q_{n-1}) = q_n(R_n q_n + q_{n-1}),$$

where $R_n q_n + q_{n-1} > q_n$, which then gives us the following inequality:

$$q_n(R_n q_n + q_{n-1}) > q_n^2. \tag{30}$$

So, from equation (30), we obtain

$$\frac{1}{q_n^2} > \left|\frac{1}{q_n(R_n q_n + q_{n-1})}\right|,$$
$$\frac{1}{q_n^2} > \left|x_n - \frac{p_n}{q_n}\right|. \tag{31}$$

Thus, from equation (28) and equation (31), we obtain that

$$\left|x_n - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

as desired. ■

In Theorem 4, we showed in general that the convergent $\frac{p_{n+2}}{q_{n+2}}$ provides a better approximation than $\frac{p_n}{q_n}$. Thus, each successive convergent provides a better approximation. Hence, the next property of θ-convergent is also true in general, and the proof is shown in the next theorem.

Theorem 6. *Let $x = \left[a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots, a_n\theta + \frac{1}{R_n}\right]$ be an irrational number with θ-convergent $\frac{p_n}{q_n}$, where $R_n = x_{n+1}$, $p_n = a_n\theta p_{n-1} + p_{n-2}$, and $q_n = a_n\theta q_{n-1} + q_{n-2}$. For every $n \geqslant 0$, the convergents are successively closer to x_n in the sense that*

$$\left|x_n - \frac{p_n}{q_n}\right| < \left|x_n - \frac{p_{n-1}}{q_{n-1}}\right|. \tag{32}$$

Proof. From equation (5), we have

$$x_n = \frac{R_n p_n + p_{n-1}}{R_n q_n q_{n-1}}.$$

So, if we subtract x_n from $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$, we obtain the following:

$$\left|x_n - \frac{p_n}{q_n}\right| = \left|\frac{R_n p_n + p_{n-1}}{R_n q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{1}{q_n(R_n q_n + q_{n-1})}\right|. \tag{33}$$

$$\left|x_n - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{R_n p_n + p_{n-1}}{R_n q_n + q_{n-1}} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{1}{q_{n-1}(R_n q_n + q_{n-1})}\right|. \tag{34}$$

Then, we compare the denominators of equations (33) and (34), we have

$$q_n(R_n q_n + q_{n-1}) > q_{n-1}(R_n q_n + q_{n-1}) \tag{35}$$

since from the behaviour of q_n, we have $q_n > q_{n-1}$. So, from equation (35), we obtain

$$\left|\frac{1}{q_{n-1}(R_n q_n + q_{n-1})}\right| > \left|\frac{1}{q_n(R_n q_n + q_{n-1})}\right|. \tag{36}$$

Next, by substituting equations (33) and (34) into equation (36), we obtain

$$\left|x_n - \frac{p_{n-1}}{q_{n-1}}\right| > \left|x_n - \frac{p_n}{q_n}\right|. \tag{37}$$

Finally, rearranging equation (37), we obtain

$$\left|x_n - \frac{p_n}{q_n}\right| < \left|x_n - \frac{p_{n-1}}{q_{n-1}}\right|$$

as desired. ∎

Next subsection, we discuss the numerical computations to support the theoretical statements on the behaviours and patterns of θ-convergents.

3.2. Numerical results on θ-convergent

This research focuses on the infinite expansions that involve the computation of irrational numbers, x. To compute the value of θ-convergent, θ-Expansions Algorithm is applied where the algorithm is as follows.

Algorithm 1 θ-Expansions Algorithm.

Require: x be a real number, $x = x_0$
Ensure: θ-expansions $x = [a_0\theta, a_1\theta, a_2\theta, a_3\theta, \ldots]$ for $n \geqslant 0$ with $0 < \theta < 1$
1: Set a_n to be the integral part of x_n, such that $a_n = \lfloor x_n \rfloor$ with $n \geqslant 0 \in \mathbb{Z}$
2: **repeat**
3: Compute $x_n - a_n$
4: **if** $x_n - a_n \neq 0$ **then**
5: Set $x_{n+1} = \frac{1}{x_n - a_n}$ and go back to Step 1 to compute a_{n+1}
6: **else**
7: Terminate the algorithm
8: **until** $x_n - a_n = 0$.
9: **for** $n \geqslant 0$ with $0 < \theta < 1$
10: Multiply each value of a_n, a_{n+1}, a_{n+2}, a_{n+3}, … obtained with θ

The value of $a_n\theta$ obtained in Algorithm 1 is computed using Maple software. Next, to obtain the value of θ-convergent, Theorem 1 and Theorem 2 are applied within the Maple software. Numerical computations on the behaviours of θ-convergent are provided as follows.

In determining the value of x and θ, they must satisfy $x \in (0, \theta)$ with $0 < \theta < 1$. We let our irrational number, $x = \{\sqrt{18}\} \approx 0.242640686$ with two different values of θ, which are $\theta_1 = \{\sqrt{21}\} \approx 0.582575695$ and $\theta_2 \approx 0.732050808$. The computation of convergents is depicted in the following table.

Table 1. The values of convergent, C_n, on two different values of θ_n for $x = \{\sqrt{18}\} = 0.242640686$ with iteration $n = 6$.

C_n	$\theta_1 = \{\sqrt{21}\} = 0.582575695$	$\theta_2 = 0.732050808$
C_0	0	0
C_1	0.4291287847	0.3415063507
C_2	0.3929478168	0.3226892761
C_3	0.3957611307	0.3236719552
C_4	0.3955409330	0.3236204891
C_5	0.3955581591	0.3236231842
C_6	0.3955564076	0.3236229993

Theorem 3 states that every value of even θ-convergent is always less than every value of odd θ-convergent. Based on the above sample in Table 1, $x = \{\sqrt{18}\} = 0.242640686$ with $\theta = \{\sqrt{21}\} = 0.582575695$, we observe the following results.

When n is odd:

- $C_1 = 0.4291287847 > C_0 = 0$;
- $C_3 = 0.3957611310 > C_2 = 0.3929478171$;
- $C_5 = 0.3955581593 > C_4 = 0.3955409333$.

From the above inequalities, every value of odd θ-convergent is greater than every value of even θ-convergent.

When n is even:

- $C_2 = 0.3929478171 < C_1 = 0.4291287847$;
- $C_4 = 0.3955409333 < C_3 = 0.3957611310$;
- $C_6 = 0.3955564079 < C_5 = 0.3955581593$.

From the above inequalities, every value of even θ-convergent is less than every value of odd θ-convergent, which satisfies $C_1 > C_0$, $C_3 > C_2$, $C_5 > C_4$, and $C_2 < C_1$, $C_4 < C_3$, $C_6 < C_5$. Thus, these clarifications lead to the following properties of θ-convergents:

$$C_0 = 0 < C_2 = 0.3929478171 < C_4 = 0.3955409333 < C_6 = 0.3955564079$$
$$< C_5 = 0.3955581593 < C_3 = 0.3957611310 < C_1 = 0.4291287847.$$

When simplifying the above inequalities, we obtain the fundamental result:

$$C_0 < C_2 < C_4 < C_6 < C_5 < C_3 < C_1.$$

In addition, the following table illustrates the other patterns of odd θ-convergent and even θ-convergent.

Table 2. The values of odd θ-convergent and even θ-convergent for $x = \{\sqrt{18}\} = 0.242640686$ with $\theta = \{\sqrt{21}\} = 0.582575695$.

n	**Odd θ-C_n**	n	**Even θ-C_n**
1	0.4291287847	2	0.3929478171
3	0.3957611310	4	0.3955409333
5	0.3955581593	6	0.3955564079

Table 2 illustrates that, based on this sample, odd θ-convergent forms a strictly decreasing sequence, while the even θ-convergent forms a strictly increasing sequence. These numerical results satisfy Theorem 4.

Now, let us compare the value of θ-convergent to the value of the irrational number x to see how good an approximation each one provides. At first, we tabulate the following table.

Table 3. The values of C_n and $|x - C_n|$ for $x = \{\sqrt{18}\} = 0.242640686$ with $\theta = 0.732050808$.

n	C_n	$\lvert x - C_n\rvert$
0	0	0.242640686
1	0.3415063507	0.0988656647
2	0.3226892761	0.0800485901
3	0.3236719552	0.0810312692
4	0.3236204891	0.0809798031
5	0.3236231842	0.0809824982

The value of $|x - C_n|$ in Table 3 illustrates the quality of each approximation. Notice, for instance, that the error, when $n = 0$ in the approximation given by $C_0 = 0$ is within the bound of

$$\frac{1}{q_0 q_1} = \frac{1}{1 \cdot 2.928203232} = 0.3415063507,$$

in which $|x - C_0| = 0.242640686$ always lies between $-\frac{1}{q_0 q_1} = -0.3415063507$ and $+\frac{1}{q_n q_{n+1}} = 0.3415063507$, satisfying Theorem 5.

4. Conclusion

Throughout this paper, we have provided an overview of convergents of θ-expansions. We can regard an infinite θ-expansion, $x = [a_0\theta; a_1\theta, a_2\theta, \ldots]$, as an analog of an infinite decimal expansion of real numbers. $C_n = \frac{p_n}{q_n}$ denotes the nth convergent of this θ-expansion, known as θ-convergent. θ-convergents are divided into two types: the even θ-convergents and the odd θ-convergents. These θ-convergents have been computed based on the θ-expansion algorithm in Maple software. Maple software helped to produce accurate θ-convergent outputs for the irrational number $x \in \mathbb{R}$ within the lowest computational time.

The even θ-convergents, $C_0, C_2, C_4, \ldots$ form a strictly increasing sequence of real numbers, while the odd θ-convergents, $C_1, C_3, C_5, \ldots$ form a strictly decreasing sequence of real numbers. Consequently, every value of an even θ-convergent is less than every value of an odd θ-convergent, summarized as $C_0 < C_2 < C_4 < \ldots < C_5 < C_3 < C_1$. These properties of θ-convergents, together with the property in Theorem 5, allow us to determine how good the approximation for each θ-convergent. The error in the approximation is bounded by $\frac{1}{q_n q_{n+1}}$ as given in Theorem 5. Furthermore, Theorem 4 has shown in general that the θ-convergent $\frac{p_{n+2}}{q_{n+2}}$ provides a better approximation than $\frac{p_n}{q_n}$. Each successive θ-convergent provides a better approximation. Hence, the property in Theorem 6 is also true and is proved in general.

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Поведінка збіжних дробів у θ-розвиненнях: обчислювальний аналіз на основі алгоритму θ-розвинень з використанням програмного забезпечення Maple

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Неперервні дроби природно виникають у діленні в стовпчик та теорії наближення дійсних чисел раціональними числами. У цьому дослідженні розглядається реалізація збіжним дробом θ-розвинення дійсних чисел $x \in (0, \theta)$ з $0 < \theta < 1$. Збіжні дроби θ-розвинення також називаються θ-збіжними дробами розвинення неперервних дробів. Метою цього дослідження є встановлення властивостей для сім'ї θ-збіжних дробів у θ-розвиненнях. Ідея виявлення поведінки θ-збіжного дробу виникла з концепції розширення регулярного неперервного дробу (RCF) і послідовностей, які є в θ-розвиненнях. Алгоритм θ-розвинення використовувався для обчислення значень θ-збіжних дробів за допомогою програмного забезпечення Maple. Це виявилося ефективним методом для швидкої комп'ютерної реалізації. Швидкість зростання θ-збіжного дробу була досліджена, щоб підкреслити ефективність θ-збіжної дробу. Аналіз θ-збіжного дробу виявив збіжні дроби, які дають краще наближення, в яких виникають менші похибки збіжності. У статті ретельно визначено поведінку θ-збіжного дробу, яка визначає, наскільки добре число x апроксимується своїми збіжними дробами для майже всіх ірраціональних чисел.

Ключові слова: *θ-збіжний дріб; θ-розвинення; θ-алгоритм розвинення; неперервний дріб; помилки збіжності.*