MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 4, pp. 1189–1197 (2024)

Verification algorithm for Lopatynsky condition

Ilkiv V. S., Kalenyuk P. I.

Lviv Polytechnic National University,
12 S. Bandera Str., 79013, Lviv, Ukraine

(Received 18 April 2024; Revised 25 April 2024; Accepted 2 December 2024)

Authors studied the Lopatynsky condition to single out among properly elliptic differential equations in the Douglis–Nirenberg sense those ones with given boundary conditions that generate an elliptic problem. This condition can be written in various ways, in particular, in algebraic form also. A new algebraic formulation of this condition is found and an algorithm for its verification is presented. Examples of its verification are given as well.

Keywords: *elliptic systems; Douglis–Nirenberg elliptic systems; elliptic boundary value problems; Lopatynsky condition; complementing condition; Bitsadze system; Laplace operator; Navier–Stokes system.*

2010 MSC: 35J48, 35J58 **DOI:** 10.23939/mmc2024.04.1189

1. Introduction

Elliptic theory is a classical theory of partial differential equations that has been developing over the last century. It covers elliptic equations and systems, pseudo-differential equations, overdetermined systems of equations, etc. The results of this theory are published in numerous monographs [1–10] and articles [11–14].

General boundary value problems for systems of elliptic equations were first considered in 1953 by Ya. B. Lopatynsky [15]. In addition to the ellipticity of the equations system, the condition of properly ellipticity and the condition of complementarity or the covering condition, connecting the differential expressions that define the system of differential equations and boundary conditions, are formulated.

The latter condition is called the Lopatynsky condition (one special case of the condition is simultaneously used in [16]).

This condition also arises when considering parabolic problems, nonlocal and other ones [17–21]. The algebraic formulation of the Lopatynsky condition is often used [1, 9, 11, 12].

The paper proposes a new algebraic formulation of the Lopatynsky condition. At the same time, the algorithm for verifying an elliptic problem becomes simpler and clearer.

The simplification of the Lopatynsky condition proposed in [22] is not correct, since, for example, it does not identify the Dirichlet problem for the system of Laplace equations ($\Delta u_1 = 0$, $\Delta u_2 = 0$) as elliptic one.

2. Elliptic boundary value problem

Let Ω be smooth compact manifold of dimension n with a boundary Γ, which is a smooth $n-1$-dimensional manifold without boundary, $n \geqslant 2$. Point $x \in \Omega$ is identified with a set of local coordinates $(x_1, \dots, x_n)$, in the neighbourhood of a manifold Γ the points are in the form $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$ are local coordinates on Γ.

Define $d_t = d/dt$ and $D = (D_1, \dots, D_n)$, where $D_j = -i\partial_j = -i\partial/\partial x_j$ for $j = 1, \dots, n$.

Sequences of integers $\{s_j\}_{j=1}^{p}$, $\{t_k\}_{k=1}^{p}$, $\{\sigma_j\}_{j=1}^{s}$ satisfy the conditions $t_1 \geqslant \dots \geqslant t_p \geqslant 0 = s_1 \geqslant \dots \geqslant s_p$, $\sigma_1 \geqslant \dots \geqslant \sigma_s$, where p and s are natural numbers.

Consider a boundary value problem

The work was carried out under the state budget topic VM4, No. 0121U114596.

$$Au = f \quad \text{on} \quad \Omega \setminus \Gamma, \quad Bu = g \quad \text{on} \quad \Gamma, \tag{1}$$

where u and f are height p columns of functions on Ω, and g is height s column of functions on Γ.

Matrix differential operator

$$Au = \begin{pmatrix} A_{11} & \dots & A_{1p} \\ & \dots\dots\dots & \\ A_{p1} & \dots & A_{pp} \end{pmatrix} \tag{2}$$

is set on Ω, where

$$A_{jk} = A_{jk}(x, D) = \sum_{|\alpha| \leqslant s_j + t_k} a^{\alpha}_{jk}(x) D^{\alpha} = A^0_{jk}(x, D) + \sum_{|\alpha| < s_j + t_k} a^{\alpha}_{jk}(x) D^{\alpha}, \tag{3}$$

if $s_j + t_k \geqslant 0$, and $A_{jk} = 0$, if $s_j + t_k < 0$.

Matrix differential boundary operator

$$B = \begin{pmatrix} B_{11} & \dots & B_{1p} \\ & \dots\dots\dots & \\ B_{s1} & \dots & B_{sp} \end{pmatrix} \tag{4}$$

is set on Γ, where

$$B_{jk} = B_{jk}(x, D) = \sum_{|\alpha| \leqslant \sigma_j + t_k} b^{\alpha}_{jk}(x) D^{\alpha} = B^0_{jk}(x, D) + \sum_{|\alpha| < \sigma_j + t_k} b^{\alpha}_{jk}(x) D^{\alpha}, \tag{5}$$

if $\sigma_j + t_k \geqslant 0$, and $B_{jk} = 0$, if $\sigma_j + t_k < 0$.

Coefficients a^{α}_{jk} and b^{α}_{jk} of operators A^{α}_{jk} and B^{α}_{jk} are the smooth complex functions.

Assume that the operator A (system of equations $Au = f$) is elliptic in the sense of Douglis–Nirenberg on Ω, that is

$$\det a_0(x, \xi) \neq 0, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \tag{6}$$

where $a_0(x, \xi) = \left(A^0_{jk}(x, \xi)\right)_{j,k=1,\dots,p}$ is the main part of the symbol $a(x, \xi)$ of the operator A.

Matrix $a_0(x, \xi)$ has a homogeneity property

$$a_0(x, \lambda\xi) = \begin{pmatrix} \lambda^{s_1} & & \\ & \ddots & \\ & & \lambda^{s_p} \end{pmatrix} a_0(x, \xi) \begin{pmatrix} \lambda^{t_1} & & \\ & \ddots & \\ & & \lambda^{t_p} \end{pmatrix}, \tag{7}$$

$s \times p$-matrix $b_0(x, \xi) = \left(B^0_{jk}(x, \xi)\right)_{\substack{j=1,\dots,s \\ j=1,\dots,p}}$ also has a similar homogeneity property

$$b_0(x, \lambda\xi) = \begin{pmatrix} \lambda^{\sigma_1} & & \\ & \ddots & \\ & & \lambda^{\sigma_s} \end{pmatrix} b_0(x, \xi) \begin{pmatrix} \lambda^{t_1} & & \\ & \ddots & \\ & & \lambda^{t_p} \end{pmatrix}. \tag{8}$$

Let $x \in \Gamma$, an arbitrary real unit vector τ tangent to Γ in the point x, unit vector $\nu = \nu(x)$ of the internal normal at the point x and define $a_0(z) = a_0(x, \tau + z\nu)$, $b_0(z) = b_0(x, \tau + z\nu)$.

Let us also assume that the operator A is proper elliptic on Γ, that is, the scalar operator with the symbol $\det a_0(x, \xi)$ is proper elliptic. This means that the polynomial $\det a_0(z)$ of the variable z has equal roots (taking into account multiplicity) on both sides of the real axis for all $x \in \Gamma$ and arbitrary tangent to Γ in the point x vector τ.

This implies that $\sum_{j=1}^{p}(s_j + t_j)$ is an even positive integer $2s$ and space $\mathfrak{M}^+ = \mathfrak{M}^+(x, \tau)$ of stable ($v(t) \to 0$, if $t \to +\infty$) solutions $v(t)$ on a semi-axis $t > 0$ of the ordinary differential equations system with constant coefficients $a_0(d_t)v = 0$ has a dimension s.

Definition 1. *The boundary problem* (1) *is called elliptic if the operator A is proper elliptic and the Lopatynsky condition is satisfied: at every point $x \in \Gamma$ and for every unit vector τ tangent to Γ in the point x the problem*

$$a_0(d_t)v = 0, \quad b_0(d_t)v|_{t=0} = h \tag{9}$$

is solvable for an arbitrary vector $h \in \mathbb{C}^s$ in the space $\mathfrak{M}^+ = \mathfrak{M}^+(x, \tau)$.

More precisely, according to the definition an elliptic problem (1) is called elliptic in the sense of Douglis–Nirenberg. If $s_1 = \ldots = s_p$, then the operator A is elliptic in the sense of Petrovsky, so the boundary value problem (1) is called elliptic in the sense of Petrovsky.

3. Formulation of the Lopatynsky condition

Let the matrix $V = V(t)$ is a base in space $\mathfrak{M}^+$ of stable solutions of the system of differential equations $a_0(d_t)v = 0$, where elements are the columns of this matrix, i.e. $V = (v_1, \ldots, v_s)$. The Lopatynsky condition is fulfilled if and only if, when the Lopatynsky matrix $L(x,\tau) = b_0(d_t)V|_{t=0}$ is non-degenerate, namely

$$\det L(x,\tau) = \det\big(b_0(d_t)V|_{t=0}\big) \neq 0, \quad x \in \Gamma, \quad \tau \in S^n, \quad \tau \perp \nu(x), \tag{10}$$

where S^n is a unit sphere in the space $\mathbb{R}^n$.

For the algebraic formulation, we denote the polynomial $a_0^+(z) = (z - z_1)\ldots(z - z_s)$, where $z_j = z_j(x,\tau)$ are roots (taking into account the multiplicity) of the polynomial $\det a_0(z)$, lying in the upper open complex half-plane, and a closed simple contour γ_+ in this half-plane that surrounds the roots $z_1, \ldots, z_s$.

Consider $p \times pt_1$ matrix

$$\tilde{V}(t) = \oint_{\gamma_+} e^{izt}\, a_0^{-1}(z)\begin{pmatrix}1 & z & \ldots & z^{t_1-1}\end{pmatrix} \otimes E_p\, dz,$$

where $\otimes$ is Kronecker matrix product, namely $\begin{pmatrix}1 & z & \ldots & z^{t_1-1}\end{pmatrix} \otimes E_p = \begin{pmatrix}E_p & zE_p & \ldots & z^{t_1-1}E_p\end{pmatrix}$, t_1 is the maximum (according to to Douglas–Nirenberg definition of ellipticity) degree of polynomials being the elements of the matrix $a_0(z)$, E_p is unit matrix of order p, then the space $\mathfrak{M}^+$ is the linear span of the matrix $\tilde{V}(t)$ columns.

The Lopatynsky's condition is the linear independence of the rows $s \times pt_1$-matrix

$$b_0(d_t)\tilde{V}(t)|_{t=0} = \oint_{\gamma_+} b_0(z)\, a_0^{-1}(z)\begin{pmatrix}1 & z & \ldots & z^{t_1-1}\end{pmatrix} \otimes E_p\, dz, \tag{11}$$

i.e., it has a full (maximum) rank, therefore $\operatorname{rank}\big(b_0(d_t)\tilde{V}(t)|_{t=0}\big) = s$.

Algebraic condition of the form (11) proposed by Ya. B. Lopatynsky in paper [15] for Petrovsky elliptical ($t_1 = \ldots = t_p$) equations systems (1). For elliptic Douglis–Nirenberg systems, the following condition is given in [14].

Let us define as $a^0 = a^0(z)$ adjoint matrix composed of the cofactors of the corresponding elements of the matrix a_0, for which the formula $a_0a^0 = \det a_0 \cdot E_p$ is true. Then the matrix V from (10) can be rewritten in the integral form

$$V(t) = \oint_{\gamma_+} a^0(z)\, H(z)\frac{e^{izt}}{a_0^+(z)}\, dz,$$

where $H(z)$ is some polynomial $p \times s$-matrix with power less than t_1, and condition (10) is the non-degeneracy of the matrix $\oint_{\gamma_+} b_0(z)\, a^0(z)\, H(z)\, \frac{dz}{a_0^+(z)}$, that is

$$\det \oint_{\gamma_+} Q(z)\, H(z)\, \frac{dz}{a_0^+(z)} \neq 0,$$

where

$$Q(z) = b_0(z)\, a^0(z) \tag{12}$$

is a rectangular $s \times p$ matrix.

This leads to the following algebraic formulation of the Lopatynsky condition:

Rows of the matrix $Q(z)$ are linearly independent modulo polynomial $a_0^+(z)$. (13)

This means that, out of equality $CQ(z) = C(z)a_0^+(z)$, where C is complex s-dimensioned vector, $C(z)$ is complex p-dimensioned polynomial vector, equality $C = 0$ follows.

In the case, when polinomial matrix $R(z)$ are the remainders from division matrix $Q(z)$ by a polynomial $a_0^+(z)$, then the Lopatynsky condition is as follows:

$$\text{Rows of matrix } R(z) \text{ are linearly independent.} \tag{14}$$

In scalar case of the problem (1) matrix V is

$$V(t) = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{\big(a_1^+(z) \dots a_s^+(z)\big)}{a_0^+(z)} e^{izt}\, dz,$$

where $a_0^+(z) = q_0 z^s + q_1 z^{s-1} + \dots + q_s$ and $a_j^+(z) = q_0 z^{s-j} + q_1 z^{s-j-1} + \dots + q_{s-j}$ for $j = 1, \dots, s$.

The Lopatynsky's condition takes the form

$$\det\big(b_0(d_t)\, \tilde{V}(t)|_{t=0}\big) = \frac{1}{2\pi i} \det \oint_{\gamma_+} b_0(z) \frac{\big(a_1^+(z)\ \dots\ a_s^+(z)\big)}{a_0^+(z)}\, dz \neq 0.$$

4. New form of the Lopatynsky condition and algorithm for its verification

Let the roots of a complex polynomial $\psi(z) = (z-\zeta_1)^{\alpha_1} \dots (z-\zeta_r)^{\alpha_r}$, where $\alpha_1 + \dots + \alpha_r = \alpha$, are ordered, for example, in ascending order of modules, and if the modules are equal, in ascending order of arguments, and matrix $S(z)$ is $\alpha_j - 1$ differentiated in the point ζ_j, that is, there are derivatives $S^{(\alpha_j - 1)}(\zeta_j)$ for $j = 1, \dots, r$. Let us enter the numeric matrix $M_{S(z)}\big(\psi(z)\big)$, which is called the value of the matrix $S(z)$ at the roots of the polynomial $\psi(z)$.

Definition 2. *Value of the matrix $S(z)$ at the roots of the polynomial $\psi(z)$, where $z \in \mathbb{C}$, is called the next block matrix*

$$M_{S(z)}\big(\psi(z)\big) = \left(S(\zeta_1)\, S'(\zeta_1) \dots \frac{1}{(\alpha_1 - 1)!} S^{(\alpha_1 - 1)}(\zeta_1) \dots S(\zeta_r)\, S'(\zeta_r) \dots \frac{1}{(\alpha_r - 1)!} S^{(\alpha_r - 1)}(\zeta_r)\right). \tag{15}$$

This matrix (in its transposed form) was introduced by P. S. Kazimirs'kii [23] when finding conditions for the factorization of matrix polynomials [24]. The problem of decomposition of a matrix polynomial into regular factors was formulated by Ya. B. Lopatynsky [25] and studied it in the work [26].

The following property of this matrix is obvious:

$$M_{S(z)}\big(\psi(z)\big) = \big(M_{S(z)}\big((z-\zeta_1)^{\alpha_1}\big) \dots M_{S(z)}\big((z-\zeta_r)^{\alpha_r}\big)\big).$$

If $S(z)$ is a polynomial matrix of degree α, then according to the Taylor formula

$$S(z) = \sum_{q=0}^{\alpha} \frac{1}{q!} S^{(q)}(\zeta)(z-\zeta)^q = M_{S(z)}\big((z-\zeta)^{\alpha+1}\big) \begin{pmatrix} 1 \\ (z-\zeta) \\ \dots \\ (z-\zeta)^{\alpha} \end{pmatrix} \otimes E, \tag{16}$$

where the size of the unit matrix E is equal to the number of columns of the matrix $S(z)$.

In particular, the coefficients $S(z)$ form the matrix $M_{S(z)}\big(z^{\alpha+1}\big)$ and

$$S(z) = \sum_{q=0}^{\alpha} \frac{1}{q!} S^{(q)}(0) z^q = M_{S(z)}\big(z^{\alpha+1}\big) \begin{pmatrix} 1 \\ z \\ \dots \\ z^{\alpha} \end{pmatrix} \otimes E. \tag{17}$$

Thus, the Lopatynsky condition (14) can be written as follows:

$$\operatorname{rank} M_{R(z)}\big(z^s\big) = s. \tag{18}$$

For an arbitrary polynomial $\psi(z)$ of degree $\alpha - 1$ based on (16) and (17) for the matrix $S(z)$ with p columns we have the formula

$$M_{S(z)}\big(\psi(z)\big) = M_{S(z)}\big(z^{\alpha}\big)\, W_{\alpha,p}\big(\psi(z)\big),$$

where

$$W_{\beta,p}\big(\psi(z)\big) = M_{\mathrm{col}\,(1,z,\dots,z^{\beta-1}) \otimes E_p}\big(\psi(z)\big) = W_{\beta}\big(\psi(z)\big) \otimes E_p,$$

and $W_{\beta}\big(\psi(z)\big) = W_{\beta,1}\big(\psi(z)\big)$ is Vandermond matrix size $\beta \times \alpha$ of polynomial $\psi(z)$.

The equality of ranks follows from the non-degeneracy of the Vandermonde matrix $W_{\alpha}\big(\psi(z)\big)$:

$$\operatorname{rank} M_{S(z)}\big(\psi(z)\big) = \operatorname{rank} M_{S(z)}\big(z^{\alpha}\big),$$

and condition (18) transforms as

$$\operatorname{rank} M_{R(z)}\big(a_0^+(z)\big) = s. \tag{19}$$

The remainder matrix $R(z)$ is defined by the matrix equation

$$Q(z) = \tilde{Q}(z)a_0^+(z) + R(z),$$

where $\tilde{Q}(z)$ is a polynomial matrix (an incomplete fraction), so for $q = 0, 1, \ldots, \alpha_j - 1$ and $j = 1, \ldots, r$ satisfies the conditions $R^{(q)}(z_j) = Q^{(q)}(z_j)$. That means equality $M_{R(z)}\big(a_0^+(z)\big) = M_{Q(z)}\big(a_0^+(z)\big)$ of the values of the matrices $R(z)$ and $Q(z)$ at the roots of the polynomial $a_0^+(z)$.

Finally, we get a new form of the Lopatynsky condition:

for arbitrary $x \in \Gamma$, $\tau \in S^n$ and $\tau \perp \nu(x)$ value of the matrix $Q(z)$ at the roots of polynomial $a_0^+(z)$ has linearly independent rows, that is $\operatorname{rank} M_{Q(z)}\big(a_0^+(z)\big) = s$. (20)

To test the Lopatynsky condition, we use the following algorithm:

for every vector $(x, \tau) \in \Gamma \times S^n$, where $\tau \perp \nu(x)$,

1) write out the matrix $a_0(z)$ of order p and the matrix $b_0(z)$ of size $s \times p$,
2) find the matrix $a^0(z)$ of order p,
3) calculate the matrix $Q(z) = b_0(z)a^0(z)$ of size $s \times p$,
4) define the polynomial $a_0^+(z)$ of degree s,
5) form a matrix $M_{Q(z)}\big(a_0^+(z)\big)$ of size $s \times ps$,
6) check whether the rank of the last matrix is equal to s.

If at least for one point $(x, \tau) \in \Gamma \times S^n$, where $\tau \perp \nu(x)$, the rank is not equal to s, i.e., it is smaller, then the Lopatynsky condition for the problem (1) is not satisfied; otherwise, the Lopatynsky condition is satisfied.

The new form of the Lopatynsky condition does not involve the calculation of the matrix $R(z)$, it is enough to calculate the value of the matrix $Q(z)$ at the roots of the polynomial $a_0^+(z)$.

For the scalar case (one equation) $Q(z) = b_0(z)$ and matrix $M_{Q(z)}\big(a_0^+(z)\big)$ is square, so we check the inequality $\det M_{b_0(z)}\big(a_0^+(z)\big) \neq 0$.

5. Example

There are considered a few examples of checking the Lopatynsky condition.

5.1. Failure to satisfy the Lopatynsky condition

Consider the Dirichlet problem for the Bitsadze system of equations

$$Au \equiv a_0(D)u = \begin{pmatrix} D_2^2 - D_1^2 & 2D_1D_2 \\ -2D_1D_2 & D_2^2 - D_1^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

The boundary conditions are defined by the matrix $B = b_0(D) = E_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

For this problem $n = p = s = 2$ and $b_0(\xi) = b_0(z) = E_2$, $a_0(z) = a_0(\tau + z\nu)$, $a^0(z) = a^0(\tau + z\nu)$, where $\xi = (\xi_1, \xi_2)$, $\nu = (\nu_1, \nu_2) = \nu(x) = \big(\nu_1(x), \nu_2(x)\big)$, $\tau = \big(-\nu_2(x), \nu_1(x)\big)$,

$$a_0(\xi) = \begin{pmatrix} \xi_2^2 - \xi_1^2 & 2\xi_1\xi_2 \\ -2\xi_1\xi_2 & \xi_2^2 - \xi_1^2 \end{pmatrix}, \quad a^0(\xi) = \begin{pmatrix} \xi_2^2 - \xi_1^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_2^2 - \xi_1^2 \end{pmatrix}, \quad Q(z) = b_0(z)a^0(z) = a^0(z).$$

Due to the formula

$$\det a_0(\xi) = \big(\xi_2^2 - \xi_1^2\big)^2 + 4\xi_1^2\xi_2^2 = \big(\xi_2^2 + \xi_1^2\big)^2 > 0, \quad \xi \neq 0,$$

the system is elliptical and for $\xi = \tau + z\nu = (z\nu_1 - \nu_2, z\nu_2 + \nu_1)$ equation

$$\big((z\nu_1 - \nu_2)^2 + (z\nu_2 + \nu_1)^2\big)^2 = (z^2 + 1)^2 = 0,$$

has a double solution $z = i$ in the upper complex half-plane.

Thus, $a_0^+(z) = (z - i)^2$, the matrix $M_{Q(z)}\big(a_0^+(z)\big)$ is rectangular of size 2×4 and

$$M_{Q(z)}\big(a_0^+(z)\big) = M_{a^0(z)}\big((z - i)^2\big) = \big(a^0(i)\, a^{0\prime}(i)\big),$$

$$a^{0\prime} = \frac{da^0}{dz}, \quad a^{0\prime}(\xi) = \begin{pmatrix} 2\xi_2\nu_2 - 2\xi_1\nu_1 & -2\xi_2\nu_1 - 2\xi_1\nu_2 \\ 2\xi_2\nu_1 + 2\xi_1\nu_2 & 2\xi_2\nu_2 - 2\xi_1\nu_1 \end{pmatrix}.$$

For $\xi = \xi^* = (\xi_1^*, \xi_2^*) = \tau + i\nu = (\nu_1 + i\nu_2)(i\ 1)$ calculate the next matrices $a^0(\xi^*)$ and $a^{0\prime}(\xi^*)$:

$$a^0(\xi^*) = \begin{pmatrix} 2(\nu_1+i\nu_2)^2 & -2i(\nu_1+i\nu_2)^2 \\ 2i(\nu_1+i\nu_2)^2 & 2(\nu_1+i\nu_2)^2 \end{pmatrix} = 2(\nu_1+i\nu_2)^2 \begin{pmatrix} 1 \\ i \end{pmatrix} (1\ \ -i),$$

$$a^{0\prime}(\xi^*) = \begin{pmatrix} 2\xi_2^*\nu_2 - 2\xi_1^*\nu_1 & -2\xi_2^*\nu_1 - 2\xi_1^*\nu_2 \\ 2\xi_2^*\nu_1 + 2\xi_1^*\nu_2 & 2\xi_2^*\nu_2 - 2\xi_1^*\nu_1 \end{pmatrix} = 2(\nu_1+i\nu_2)^2 \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

Hence, $a^{0\prime}(i) = -ia^0(i)$ and

$$M_{Q(z)}\big(a_0^+(z)\big) = a^0(i)(E_2\ \ -iE_2) = 2(\nu_1+i\nu_2)^2 \begin{pmatrix} 1 \\ i \end{pmatrix} (1\ \ -i\ \ -i\ \ -1).$$

Thus, the matrix $M_{Q(z)}\big(a_0^+(z)\big)$ has rank one, i.e. the proportional rows, in particular $(1\ i)M_{Q(z)}\big(a_0^+(z)\big) = 0$.

Therefore, according to the algorithm, the Dirichlet problem for the considered elliptic system of equations is not elliptic, which was first shown by Bitsadze [27] in 1948.

5.2. An elliptic problem in the sense of Douglis–Nirenberg

Study the Dirichlet problem for the Laplace equation $D_1^2u + D_2^2u = 0$.

This problem is equivalent to the Dirichlet problem for the system of first order equations

$$D_1u_2 + D_2u_3 = 0, \quad D_1u_1 - u_2 = 0, \quad D_2u_1 - u_3 = 0$$

with the marix $a_0(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{pmatrix}$ of the Douglis–Nirenberg structure.

In this case $b_0(z) = b_0(\xi) = (1\ 0\ 0)$ and $Q(z) = b_0(z)\,a^0(z)$, where $a^0(\xi) = \begin{pmatrix} 1 & \xi_1 & \xi_2 \\ \xi_1 & -\xi_2^2 & \xi_1\xi_2 \\ \xi_2 & \xi_1\xi_2 & -\xi_1^2 \end{pmatrix}$. Since $\det a_0(\xi) = \xi_1^2 + \xi_2^2 > 0$ for $\xi \neq 0$, then the system is elliptic in the Douglis–Nirenberg sense ($t_1 = 2$, $t_2 = t_3 = 1$, $s_1 = 0$, $s_2 = s_3 = -1$) and $a_0^+(z) = z - i$.

Let us calculate $M_{Q(z)}\big(a_0^+(z)\big) = M_{b_0(z)a^0(z)}(z - i)$, namely

$$M_{Q(z)}\big(a_0^+(z)\big) = (1\ \xi_1^*\ \xi_2^*) = (1\ i\,\nu_1 - \nu_2\,\nu_1 + i\,\nu_2).$$

The problem under consideration is elliptic one because

$$\operatorname{rank} M_{Q(z)}\big(a_0^+(z)\big) = 1 = s \quad \text{and} \quad t_1 + t_2 + t_3 + s_1 + s_2 + s_3 = 2 = 2s.$$

5.3. Navier–Stokes system of equations

Consider in $\Omega \subset \mathbb{R}^2$ the first boundary value problem (Dirichlet problem) for the Navier–Stokes equations

$$D_1^2u_1 + D_2^2u_1 - D_1u_3 = 0, \quad D_1^2u_2 + D_2^2u_2 - D_2u_3 = 0, \quad D_1u_1 + D_2u_2 = 0,$$

$$u_1\big|_\Gamma = g_1, \quad u_2\big|_\Gamma = g_2$$

and check its ellipticity according to Douglis–Nirenberg.

There is the Douglis–Nirenberg problem ($t_1 = t_2 = 2$, $t_3 = 1$, $s_1 = s_2 = 0$, $s_3 = -1$, $\sigma_1 = \sigma_2 = -2$) with the matrices

$$a_0(\xi) = \begin{pmatrix} \xi_1^2+\xi_2^2 & 0 & -\xi_1 \\ 0 & \xi_1^2+\xi_2^2 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}, \quad b_0(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

moreover the system is elliptical, $\det a_0(\xi) = \big(\xi_1^2+\xi_2^2\big)^2 > 0$ for $\xi \in \mathbb{R}^2 \setminus \{0\}$, and

$$a^0(\xi) = \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & (\xi_1^2+\xi_2^2)\xi_1 \\ -\xi_1\xi_2 & \xi_1^2 & (\xi_1^2+\xi_2^2)\xi_2 \\ -(\xi_1^2+\xi_2^2)\xi_1 & -(\xi_1^2+\xi_2^2)\xi_2 & (\xi_1^2+\xi_2^2)^2 \end{pmatrix}.$$

For $\xi = \tau + z\nu = (z\nu_1 - \nu_2, z\nu_2 + \nu_1)$ the equation $\big(\xi_1^2+\xi_2^2\big)^2 = 0$ is in the form $\big(z^2+1\big)^2 = 0$, that is, the solution $z = i$ (of multiplicity two) is unique in the upper complex half-plane and $a_0^+(z) = (z-i)^2$.

The rectangular 2×6 matrix $M_{Q(z)}\big(a_0^+(z)\big)$, where $Q(z)=b_0(z)a^00(z)$, $b_0(z)=b_0(\xi)$,

$$Q(\xi)=\begin{pmatrix}\xi_2^2 & -\xi_1\xi_2 & \big(\xi_1^2+\xi_2^2\big)\xi_1\\ -\xi_1\xi_2 & \xi_1^2 & \big(\xi_1^2+\xi_2^2\big)\xi_2\end{pmatrix},\quad Q'(\xi)=\begin{pmatrix}2\xi_2\nu_2 & -\xi_2\nu_1-\xi_1\nu_2 & 2z\xi_1+\big(\xi_1^2+\xi_2^2\big)\nu_1\\ -\xi_2\nu_1-\xi_1\nu_2 & 2\xi_1\nu_1 & 2z\xi_2+\big(\xi_1^2+\xi_2^2\big)\nu_2\end{pmatrix},$$

is calculated using the formula $M_{Q(z)}\big(a_0^+(z)\big)=\big(Q(i)\ Q'(i)\big)$.

For the vector $\xi=\xi^*=(\xi_1^*,\xi_2^*)=\tau+i\,\nu=(\nu_1+i\,\nu_2)(i\ 1)$, with absolute value $\sqrt{2}$, the next is true

$$Q(\xi^*)=(\nu_1+i\,\nu_2)^2\begin{pmatrix}1 & -i & 0\\ -i & -1 & 0\end{pmatrix}=(\nu_1+i\,\nu_2)^2\begin{pmatrix}1\\ -i\end{pmatrix}\big(1\ \ -i\ 0\big),$$

$$Q'(\xi^*)=(\nu_1+i\,\nu_2)^2\begin{pmatrix}\dfrac{2\nu_2}{\nu_1+i\,\nu_2} & -1 & \dfrac{-2}{\nu_1+i\,\nu_2}\\ -1 & \dfrac{2i\nu_1}{\nu_1+i\,\nu_2} & \dfrac{2i}{\nu_1+i\,\nu_2}\end{pmatrix}.$$

So, let us check the matrix rank $M_{Q(z)}\big(a_0^+(z)\big)$:

$$\operatorname{rank} M_{Q(z)}\big(a_0^+(z)\big)=\operatorname{rank}\begin{pmatrix}1 & -i & 0 & \dfrac{2\nu_2}{\nu_1+i\,\nu_2} & -1 & \dfrac{-2}{\nu_1+i\,\nu_2}\\ -i & -1 & 0 & -1 & \dfrac{2i\,\nu_1}{\nu_1+i\,\nu_2} & \dfrac{2i}{\nu_1+i\,\nu_2}\end{pmatrix}$$
$$=\operatorname{rank}\begin{pmatrix}\nu_1+i\,\nu_2 & 2\nu_2 & -(\nu_1+i\,\nu_2)\\ -i(\nu_1+i\,\nu_2) & -(\nu_1+i\,\nu_2) & 2i\,\nu_1\end{pmatrix}=2.$$

The last equalities follow from the linear dependence of the columns $\binom{1}{-i}$, $\binom{-i}{-1}$, $\binom{0}{0}$, $\frac{1}{\nu_1+i\nu_2}\binom{-2}{2i}$ and from the calculation of the determinant formed from the second and third columns of the 2×3 matrix:

$$\begin{vmatrix}2\nu_2 & -(\nu_1+i\,\nu_2)\\ -(\nu_1+i\,\nu_2) & 2i\,\nu_1\end{vmatrix}=4i\,\nu_1\nu_2-(\nu_1+i\,\nu_2)^2=2i\,\nu_1\nu_2-1\neq 0.$$

It is verified that the problem for Navier–Stokes system under consideration satisfies the Lopatynsky condition and is therefore an elliptic problem according to Douglis–Nirenberg.

6. Conclusions

This paper proposes a new algebraic form of the Lopatynsky condition for the systems of elliptic type partial differential equations (properly elliptic systems) in the Douglis–Nirenberg sense. It is shown how it is related to the known forms of the Lopatynsky condition and a corresponding algorithm for computing it is presented.

The algorithm for checking the ellipticity of a problem for proper elliptic systems is demonstrated on the examples of the Dirichlet problem for the Bitsadze system, the Douglis–Nirenberg system generated by the Laplace operator, and the Navier–Stokes system. In the first case, the Lopatynsky condition is not satisfied, so the problem is not elliptic, the problems are elliptic in the other cases. The algorithm consists of constructing the value of matrix at the roots of the polynomial and determining the completeness of the rank of these value.

[1] Agranovich M. S. Elliptic Boundary Problems. In: Agranovich M. S., Egorov Y. V., Shubin M. A. (eds) Partial Differential Equations IX. Encyclopaedia of Mathematical Sciences. **79**, 1–144 (1997).

[2] Berezanskii Ju. M. Expansions in Eigenfunctions of Selfadjoint Operators. Translations of Mathematical Monographs. **17** (1968).

[3] Hörmander L. Linear Partial Differential Operators. Springer, Heidelberg (1969).

[4] Hörmander L. The Analysis of Linear Partial Differential Operators. **1–4**, Springer-Verlag, Berlin (1983–85).

[5] Lions J.-L., Magenes E. Problèmes aux Limites non Homogènes et Applications. **1**. Dunod, Paris (1968).

[6] Mikhailets V. A., Murach A. A. Hörmander Spaces, Interpolation, and Elliptic Problems. De Gruyter Studies in Mathematics. **60**, De Gruyter, Berlin (2014).

[7] Miranda C. Equazioni alle Derivate Parziale di Tipo Ellyptico. Springer-Verlag, Berlin (1955).

[8] Panich O. I. Introduction to the General Theory of Elliptic Boundary-Value Problems. Kiev, Vyshcha Shkola (1986).

[9] Roitberg Ya. Elliptic Boundary Value Problems in the Spaces of Distributions. Kluwer Academic Publishing (1996).

[10] Wloka Jo. T., Rowley B., Lawruk B. Boundary Value Problems for Elliptic Systems. Cambridge University Press (1995).

[11] Agmon S., Douglis A., Nirenberg L. Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions II. Communications on Pure and Applied Mathematics. **17** (1), 35–92 (1964).

[12] Solonnikov V. A. General boundary value problems for systems elliptic in the sense of A. Douglis and L. Nirenberg. I. Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya. **28** (3), 665–706 (1964).

[13] Volevich L. R. On the Theory of Boundary Value Problems for General Elliptic Systems. Doklady Akademii Nauk SSSR. **148**, 489–492 (1963).

[14] Volevich L. R. Solvability of Boundary Value Problems for General Elliptic Systems. Mat. Sb. **68 (110)** (3), 373–416. English transl.: Am. Math. Soc. Transl., Ser. II, **61**, 182–225 (1968).

[15] Lopatinskij Ya. B. A method of reduction of boundary-value problems for systems of differential equations of elliptic type to a system of regular integral equations. Ukrainian Mathematical Journal. **5**, 123–151 (1953).

[16] Shapiro Z. Ya. On General Boundary Value Problems for Equations of Elliptic Type. Izvestia AN SSSR. **17** (6), 539–562 (1953).

[17] Agranovich M. S., Vishik M. I. Elliptic Problems with a Parameter and Parabolic Problems of General Type. Russian Mathematical Surveys. **19** (3), 53–157 (1964).

[18] Eidelman S. D., Zhitarashu N. V. Parabolic Boundary–Value Problems. Operator Theory: Advances and Applications. **101** (1998).

[19] Èskin G. Boundary Value Problems for Elliptic Pseudodifferential Equations. Translations of Mathematical Monographs. **52**, AMS, Providence (1981).

[20] Skrypnik I. V. Methods of Analysis of Nonlinear Elliptic Boundary Value Problems. Translations of Mathematical Monographs. **139**, AMS, Providence (1991).

[21] Solonnikov V. A. On Boundary Value Problems for Linear Parabolic Systems of Differential Equations of General Form. Trudy Matematicheskogo Instituta imeni V. A. Steklova. **83**, 3–163 (1965).

[22] Krupchyk K., Tuomela J. The Shapiro–Lopatinskij Condition for Elliptic Boundary Value Problems. LMS Journal of Computation and Mathematics. **9**, 287–329 (2006).

[23] Kazimirs'kii P. S. To the Decomposition of a Polynomial Matrix into Linear Factors. Dopovidi AN URSR. **4**, 446–448 (1964).

[24] Petrychkovych V. M. Generalized Equivalence of Matrices and its Collections and Factirization of Matrices over Rings. L'viv, Pidstryhach Inst. for Appl. Probl. of Mech. and Math. of the NAS of Ukraine (2015).

[25] Lopatinskij Ya. B. Decomposition of a Polynomial Matrix into a Product. Nauchn. Zap. Polytechn. Inst., L'vov, Ser. Fiz. Mat. **2**, 3–7 (1956).

[26] Lopatinskij Ya. B. On Some Properties of Polynomial Matrices. Boundary Value Probl. of Math. Phys., Kiev, Inst. of Math. AN USSR. 108–146 (1979).

[27] Bitsadze A. V. Boundary Value Problems for Elliptic Equations of Second Order. North-Holland, Amsterdam (1968).

Алгоритм перевірки умови Лопатинського

Ільків В. С., Каленюк П. І.

Національний університет "Львівська політехніка",
вул. С. Бандери, 12, 79013, Львів, Україна

У роботі досліджується умова Лопатинського, яка використовується для виділення серед правильно еліптичних диференціальних рівнянь в сенсі Дуґліса–Ніренберґа і заданих крайових умов таких, що породжують еліптичну задачу. Ця умова записується у різних варіантах, зокрема в алгебричному вигляді. Знайдено нове алгебричне формулювання цієї умови і подано алгоритм її перевірки. Наведено приклади її застосування.

Ключові слова: *еліптичні системи; еліптичні системи в сенсі Дуґліса–Ніренберґа; еліптичні крайові задачі; умова Лопатинського; умова доповнювальності; система Біцадзе; оператор Лапласа; система Нав'є–Стокса.*