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Dynamics of a fishery with nonlinear harvesting: control, price variation, and MSY

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In this paper, we construct and analyse a new fishing mathematical model, which describes the time evolution of a fish stock, which is harvested by a fishing fleet, described by its fishing effort. We consider that the price, which is given by the difference between supply and demand, is varying with respect to time. For the harvesting function, we use the Holling II function. On the other hand, we consider two different time scales: a fast one for the price variation and a slow one for fish stock and fishing effort variations. We use an "aggregation of variables" method to get the aggregated model that governs fish biomass and fishing effort in the slow time. By analyzing this reduced model, and under some conditions, we prove that three interesting equilibria can occur. Furthermore, we show how one can control the model to avoid the undesirable situations and to reach the stable equilibrium. Another interesting aspect given in this manuscript is the possibility of the implementation of Marine Protected Areas (MPAs). We show how that MPAs permits us to contribute significantly to the rehabilitation of depleted fish populations. This is achieved by disrupting the state of "Fish Extinction" equilibrium, and establishing a stable one.

Keywords: *fishery model; Holling II function; varying price; aggregation of variables; equilibria; stability; control; marine protected area; MSY.*

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1. Introduction

Many bio-economic models have been created to describe the economic and biologic aspects of the dynamics of fishery [1]. Furthermore, elementary models suppose that the population of fish grows in a logistic way, and usually assumed a catch term that is relative to the fish and to harvesting vessels. Thus, we consider, in the model, another equation which describes the fishing effort variations. This equation is represented by the difference between the net income (i.e. the catch function proportional to the market price of the resource) and the costs of the harvesting effort 'c'. This leads to the most common predator–prey fishery models [2, 3].

On the other hand, in many mathematical fishing models, the resource's market price is supposed to be constant [4, 5]. In other works, the price is not fixed and is determined by the catch-function, the fish biomass, or the number of fishing fleets [6]. That is why it became necessary to add a third equation, concerning the price variation with respect to time, which is generally represented by the difference between the amount caught (the supply) and the demand [7].

In several works [8–11], authors assumed a linear famous catch-function, it is a Schaefer function [12], i.e. the catch-ability is directly proportional to both the fish biomass and the fishing effort. The reason there exist many restrictions to this catch-function; the Schaefer function increases in a limited way with fishing effort, 'E' for a fixed fish biomass, 'n'. Authors in [8, 10] studied the catch with a varying price, they assume a linear demand function $D(p) = A - \alpha p$ in this situation, when the price becomes higher, the demand is close to zero, that means when the price on the market become very expensive there is no demand. In [13] they studied the case of a network of patches connected, with a linear harvesting which corresponds to the Schaefer function [12] but with a nonlinear demand i.e. $D(p) = \frac{A}{p}$, in this state, even if the price is very high, the demand tend to a positive value. The main result was the possibility of an over-exploitation equilibrium causing fish extinction. In [14] authors prolonged the last model to include a nonlinear harvesting function, i.e. $h(t) = \frac{qEn}{n+D}$ but the demand function used supposed linear; $D(p) = A - \alpha p$. Under certain conditions, it is possible to achieve sustainable fishing.

In the present paper, we integrate two main approaches, which means; we consider a non-linear harvesting function, relative to Holling type II, i.e. $h(t) = \frac{qEn}{n+D}$ (Figure 1). This harvesting function is more realistic than a linear one, which is usually presented as a Schaefer function, we can see clearly that there exist many limitation of the linear harvesting function such as it suppose that the harvesting vessels can harvest the stock unbounded quantity of fish stock. We also consider an hyperbolic demand function; $D(p) = \frac{A}{1+\beta p}$. These two aspects will be illustrated by a mathematical model that combines the fish and the fishing efforts evolution, and the price variation. We will focus on controlling our model to prevent undesirable cases and showing that we can implement Marine Protected Areas (MPAs) to rebuild fish stocks and prevent over-fishing.

The manuscript is structured as follows. Section 2 is dedicated to describing a mathematical model that is represented by three ordinary differential equations. Due to the presence of varying time scales, we can derive a reduced model. In Section 3, we analyze the aggregated model and present numerical simulations. Section 4 is dedicated to introducing a control parameter in the system which help us to avoid the Fish Extinction case. In Section 5, we study the impacts of installation of Marine Protected Areas (MPAs), on the fish stock dynamics. The last section of our study investigates the surface size area that results in the optimal capture at equilibrium.

2. Mathematical model with nonlinear price equation

2.1. Complete model

Here we consider a three-dimensional model governing three main variables; the fish stock which is denoted by n, the harvesting effort noted by E, and the price on the market cited by p. The fish grows and the harvesting effort occurs at a slower time scale, as compared with the price variation which follows a faster one, $\tau = \frac{t}{\varepsilon}$.

According to these assumptions, we obtain:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}\tau} = \varepsilon \left[rn\left(1 - \dfrac{n}{k}\right) - \dfrac{qnE}{n+D} \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E \left(-c + p\dfrac{qn}{n+D} \right), \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha \left(D(p) - \dfrac{qnE}{n+D} \right). \end{cases} \tag{1}$$

The first equation of the system describes the growth of fish stock in time. If we neglect the fishing activity, the fish biomass follows a logistic function in its growth. Where the positive constants k and r represent the carrying capacity and the intrinsic growth rate of the stock respectively. The fish biomass is targeted by a non-linear harvesting function, which is relative to a Holling type II i.e. $\frac{qnE}{n+D}$ is then proportional to the harvesting effort E, the catch-ability q, the fish biomass size n, and with D is the half-saturation level of stock (see Figure 1).

The second remaining equation describes the gap between the net income and the cost of harvesting fleet c. The fish price dynamics (the third equation). In our work, we assumed that the price varies according to a monotonically decreasing hyperbolic function i.e. $D(p) = \frac{A}{1+\beta p}$ where the demand achieves a maximum A when the price equal zero and the parameter β determines the degree of responsiveness of the demand to changes in price, and it is always a positive value (see Figure 2).

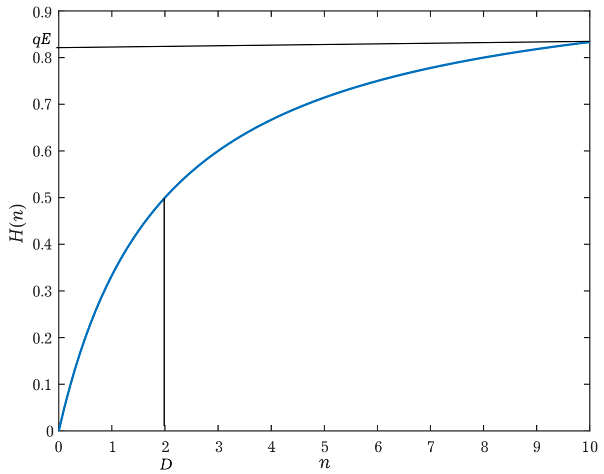

Fig. 1. The curve of Holling type 2 function for $E = 1$, $q = 0.8$, $D = 2$.

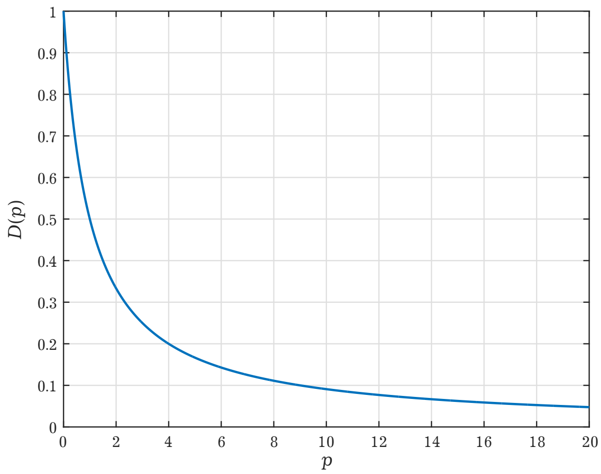

Fig. 2. The curve of the nonlinear demand function for $A = 1$ and $\beta = 1$.

2.2. Derivation of an aggregated model

We consider that the market price progresses more rapidly, and we utilizes methods of aggregation. Our model can be reduced to a model with two global variables n and E.

The price in the catch in the second equation of (1) is replaced with the solutions of the following equation (see Appendix 1):

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha\left(\frac{A}{1+\beta p} - \frac{qnE}{n+D}\right) = 0. \tag{2}$$

We get one solution of this equation, p^* given by

$$p^* = \frac{A(n+D) - qnE}{\beta qEn}. \tag{3}$$

Therefore, the second equation is represented as:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -cE + \frac{A(n+D) - qnE}{\beta(n+D)}.$$

2.3. Aggregated model

The reduced model is obtained by replacing the price (3) into the complete system (1).

Consequently, we obtain the stable model:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n\left[r\left(1-\dfrac{n}{k}\right) - \dfrac{qE}{n+D}\right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = -cE + \dfrac{A(n+D) - qnE}{\beta(n+D)}. \end{cases} \tag{4}$$

3. Analysis of the aggregated model

3.1. Existence of equilibria

The n-nullclines are $n = 0$ and $E = \frac{r}{qk}(k-n)(n+D)$, and E-nullclines are: $E = \frac{A(n+D)}{c\beta(n+D)+qn}$ (see Appendix 3), we have a point of equilibrium at the coordinates: $\left(0, \frac{A}{c\beta}\right)$, we also have the interior equilibria point (n^*, E^*) which solve the following system:

$$\begin{cases} E(n) = \dfrac{r}{qk}(k-n)(n+D), \\ E(n) = \dfrac{A(n+D)}{c\beta(n+D)+qn}. \end{cases} \tag{5}$$

We will look for the existence of positive interior equilibria of system (4). Which are solutions of (5), n^* is a positive solution of this equation

$$F(n) = n^2(rc\beta + rq) + n(rc\beta D - rc\beta k - rqk) + (Aqk - rc\beta kD).$$

The derivative of F read

$$F'(n) = r(\beta c + q)\left[2n - k + \frac{\beta c D}{\beta c + q}\right].$$

The number of fixed points of (4) is represented by the number of positive solutions of $F(n) = 0$. The following Theorem proves the existence of equilibria.

Theorem 1. *System (4) may have up to two positive interior equilibria.*

▶ *Case 1: If* $\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right) > 0$*:*

- *If* $F(0) < 0$ *so equation* $F(n) = 0$ *have a single positive solution* (n^*, E^*)*.*
- *If* $F(0) > 0$ *and* $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) < 0$*, then equation* $F(n) = 0$ *have two positive solution* (n_1^*, E_1^*) *and* (n_2^*, E_2^*) *verifying* $n_1^* < \frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right) < n_2^*$*.*
- *If* $F(0) > 0$ *and* $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) > 0$*, then equation* $F(n) = 0$ *do not have positive solutions.*

▶ *Case 2: If* $\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right) \leqslant 0$*:*

- *If* $F(0) < 0$*, so equation* $F(n) = 0$ *have a single positive solution* (n^*, E^*)*.*
- *If* $F(0) > 0$ *and* $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) < 0$*, then equation* $F(n) = 0$ *do not have positive solutions.*

Proof. See Appendix 3. ■

3.2. Analysis of local stability

The Jacobian matrix reads

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{k} - qD\frac{E}{(n+D)^2} & \frac{-qn}{n+D} \\ -\frac{qED}{\beta(n+D)^2} & -c - \frac{qn}{\beta(n+D)} \end{pmatrix}. \tag{6}$$

• At $E_0 = \left(0, \frac{A}{c\beta}\right)$:

$$Jac_{\left(0,\frac{A}{c\beta}\right)} = \begin{pmatrix} r - \frac{qA}{cD\beta} & 0 \\ \frac{qA}{cD\beta^2} & -c \end{pmatrix}.$$

The eigenvalues at $\left(0, \frac{A}{c\beta}\right)$ are: $\lambda_1 = \left(r - \frac{qA}{cD\beta}\right) - c$ and $\lambda_2 = -c\left(r - \frac{qA}{cD\beta}\right)$.

If $c < \frac{qA}{rD\beta}$, the equilibrium $E_0 = \left(0, \frac{A}{c\beta}\right)$ is stable, if not is a saddle point.

• At $E^* = (n^*, E^*)$ (represented by E_1 and E_2):

$$Jac_{(n^*,E^*)} = \begin{pmatrix} -\frac{rn^*}{k} + \frac{qn^*E^*}{(n^*+D)^2} & \frac{-qn^*}{n^*+D} \\ -\frac{qDE^*}{\beta(n^*+D)^2} & -(c + \frac{qn^*}{\beta(n^*+D)}) \end{pmatrix}.$$

Proposition 1.

▶ **Case 1:** If $\frac{1}{2}(k - \frac{\beta c D}{\beta c+q}) > 0$:

1.1 If $F(0) < 0$, then equation $F(n) = 0$ have a single positive solution with a determinant which positive, so (n^*, E^*) is stable while $\left(0, \frac{A}{c\beta}\right)$ is a saddle point.

1.2 If $F(0) > 0$ and $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) < 0$, then equation $F(n) = 0$ have two positive solution (n_1^*, E_1^*) and (n_2^*, E_2^*) verifying $n_1^* < \frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right) < n_2^*$ with (n_1^*, E_1^*) is saddle, (n_2^*, E_2^*) is stable and $\left(0, \frac{A}{c\beta}\right)$ is stable.

1.3 If $F(0) > 0$ and $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) > 0$, then equation $F(n) = 0$ do not have positive solutions.

▶ **Case 2:** If $\frac{1}{2}(k - \frac{\beta c D}{\beta c+q}) \leqslant 0$:

2.1 If $F(0) < 0$, so equation $F(n) = 0$ have a single positive solution (n^*, E^*) which stable.

2.2 If $F(0) > 0$ and $F\left(\frac{1}{2}\left(k - \frac{\beta c D}{\beta c+q}\right)\right) < 0$, then equation $F(n) = 0$ do not have positive solutions.

Proof. See Appendix 4. ■

4. Discussion of the results and numerical simulations

Figures 3 and 4 illustrate the case of the FEE which is stable if and only if $F(0) > 0$ i.e. $(c < \frac{Aq}{rD\beta})$ and that means, when c decreases, it will lead to a rise in fishing activity, which is equivalent to a downward trend in biomass until the extinction of the species. At equilibrium the fish species become extinct, and the fishing fleet converges to a positive value $\frac{A}{c\beta}$, when we approach to equilibrium the price tends to a higher value (see Figure 7). This situation aligns with the fish extinction.

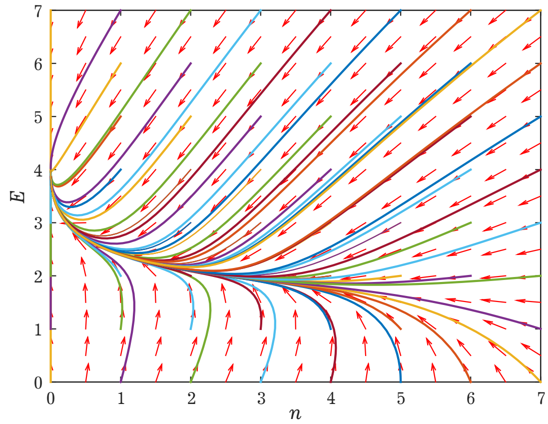

Fig. 3. Phase plan for the case of FEE, with parameters set: $c = 1$, $q = 1.7$, $A = 4$, $r = 1$, $D = 2$, $k = 5$, $\beta = 1$.

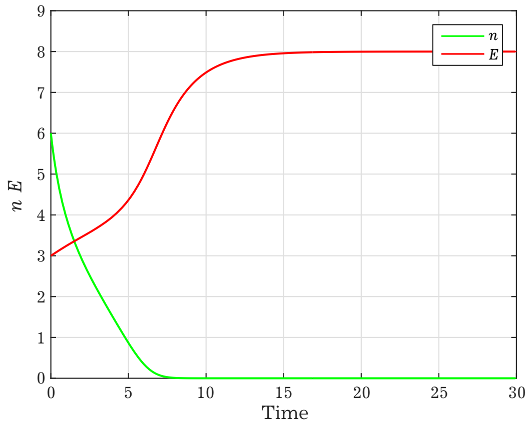

Fig. 4. Illustration of the FEE with parameters set: $c = 0.5$, $q = 1$, $A = 4$, $r = 1$, $D = 2$, $k = 5$, $\beta = 1$, $n(0) = 6$, $E(0) = 3$.

Figure 5 describes the subcases 1.1 and 2.1 showing the variations of fish biomass and fishing fleet in time. When $F(0) < 0$ means that the cost of fishing effort increases, it will lead to a decrease in fishing effort, which is equivalent to a high trend in biomass. At equilibrium, the fish population becomes higher and higher due to the high taxation on boat owners, and fishing efforts start to decrease. As a result, the resource can recover. This case corresponds to sustainable and durable fisheries. Figure 6 illustrates that any trajectory initiating in the positive quadrant converges to the sustainable fishery equilibrium (n^*, E^*).

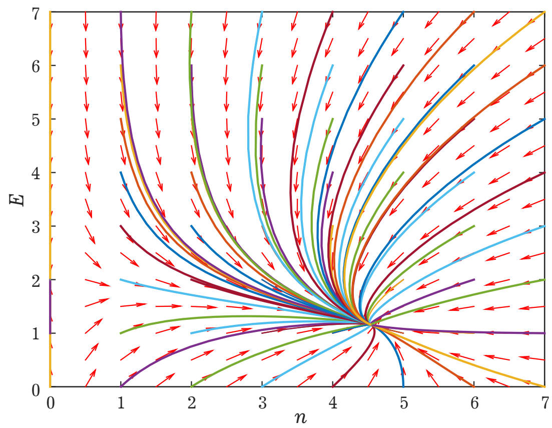

Fig. 5. Phase plan for the subcases 1.1 and 2.1, with parameters set: $c = 0.5$, $q = 1$, $A = 1$, $k = 5$, $r = 1$, $D = 8$, $\beta = 1$.

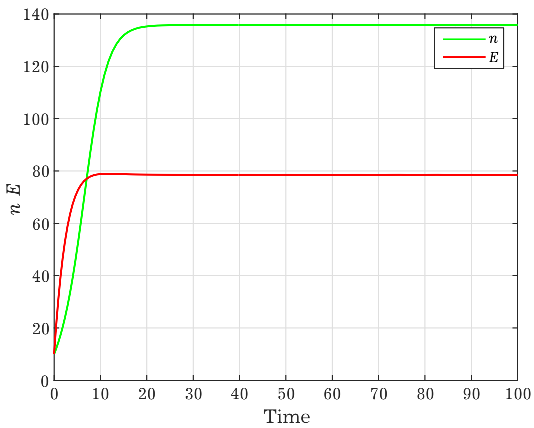

Fig. 6. Illustration of the interior equilibrium with parameters set: $c = 0.3$, $q = 0.1$, $A = 30$, $k = 150$, $r = 0.5$, $D = 30$, $\beta = 1$, $n(0) = 10$, $E(0) = 10$.

Subcases 1.2 demonstrate the existence of two positive equilibria (n_1^*, E_1^*) and (n_2^*, E_2^*) with $n_1^* < n_2^*$. The point $\left(0, \frac{A}{\beta c}\right)$ is a locally asymptotically stable (l.a.s.) equilibrium. The equilibrium with the small biomass is a saddle point while (n_2^*, E_2^*) is locally asymptotically stable. In this case, there exists a separatrix (see Figure 8); depending on the starting conditions, the trajectory will either tend towards $\left(0, \frac{A}{\beta c}\right)$ or towards the fishery equilibrium (n_2^*, E_2^*). In simpler terms, either the fish biomass will become extinct with a rise in price, or a sustainable fishery will be reached.

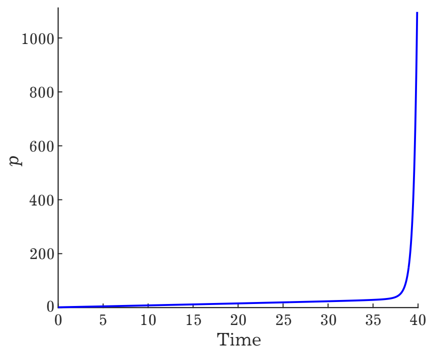

Fig. 7. Price vs time.

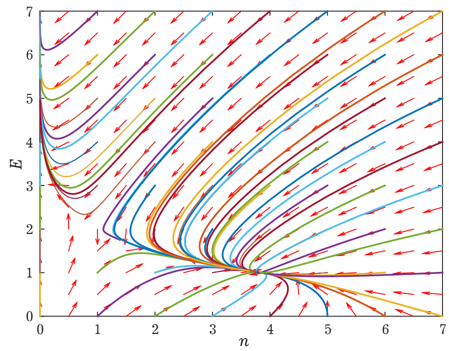

Fig. 8. Illustration of the subcase 1.2 (separatrix) with parameters set: $c = 0.1$, $q = 1.6$, $A = 1$, $k = 5$, $r = 1$, $D = 3$, $\beta = 1$.

5. Introduction of a control parameter

As demonstrated in the preceding section, depending on the values of certain parameters, the system dynamics can result in either a sustainable equilibrium or a state of fish extinction. To avoid this situation, it is preferable to minimize significant fluctuations in the overall fish stock and fishing effort.

Thus, it would be advantageous to incorporate a control parameter into the model. This was proposed in the context of spatial fisheries, [11]. This parameter, denoted as "u", is a real constant that satisfies the condition: $0 < u < 1$. One straightforward approach that a coastal state can adopt to manage its fishery is to regulate the technical capabilities of boats, such as by imposing restrictions on the types of fishing techniques that can be employed or by limiting the overall catch of fishing fleets. A reduction in the technical capacities of vessels could lead to a decrease in their catch-ability. To account for this, we introduce a catch-ability term denoted as "u", which is uniform across the entire fishing fleet. To implement this, we multiply the harvested terms $\frac{qEn}{n+D}$ by the parameter u in every equations of the system (1), resulting in the following system:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}\tau} = \varepsilon \left[rn\left(1 - \dfrac{n}{k}\right) - \dfrac{uqnE}{n+D} \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E \left(-c + p\dfrac{uqn}{n+D} \right), \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha \left(D(p) - \dfrac{uqnE}{n+D} \right). \end{cases} \tag{7}$$

In that case, the aggregated system is

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n \left[r\left(1 - \dfrac{n}{k}\right) - \dfrac{uqE}{n+D} \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = -cE + \dfrac{A(n+D) - uqnE}{\beta(n+D)}. \end{cases} \tag{8}$$

The study of this model is straightforward. The Jacobian matrix is given by

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{k} - uD\frac{qE}{(n+D)^2} & \frac{-uqn}{n+D} \\ -\frac{uqED}{\beta(n+D)^2} & -c - \frac{uqn}{\beta(n+D)} \end{pmatrix}. \tag{9}$$

In the case where $E_0 = \left(0, \frac{A}{c\beta}\right)$. The eigenvalues are respectively given by: $\lambda_1 = \left(r - \frac{uqA}{cD\beta}\right) - c$ and $\lambda_2 = -c\left(r - \frac{uqA}{cD\beta}\right)$.

If $c > \frac{uqA}{rD\beta}$ the equilibrium $E_0 = \left(0, \frac{A}{c\beta}\right)$ is saddle point. In order to have $c > \frac{uqA}{rD\beta}$, which ensures the existence of an unstable equilibrium $\left(0, \frac{A}{c\beta}\right)$, it is essential to keep the control parameter under a set maximum value:

$$0 < u < \frac{crD\beta}{Aq}. \tag{10}$$

Thus, when $H = \frac{crD\beta}{Aq}$ is higher than 1, (n^*, E^*) is always stable without control. Conversely, if H is less than 1, the system must be regulated by adjusting the parameter u.

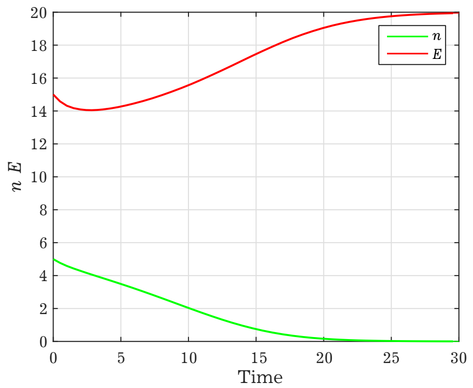

Fig. 9. Case without control of a stable Fish Extinction equilibrium with parameter values $r = 0.6$, $k = 30$, $q = 1$, $D = 20$, $c = 0.4$, $A = 8$, $n(0) = 5$, $E(0) = 15$.

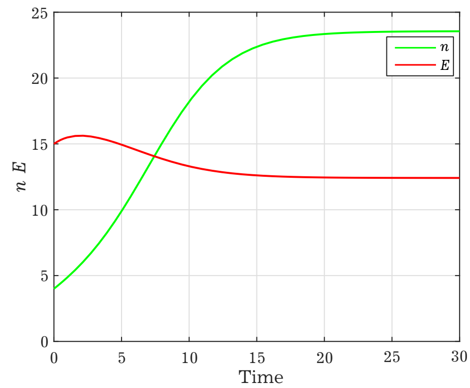

Fig. 10. Control parameter added after $t = 5$, $u = 0.6$, we notice that the control parameter destabilizes the Fish Extinction equilibrium and converges to the regular state of the system.

We now, we will take a look at a case where we have a Fish Extinction Equilibrium, Figure 9 without adding a control parameter. In Figure 10, we add a control 'u' to our system after a time interval t, to maintain a sustainable dynamic equilibrium in the system, and thus avoid the fish extinction case. We can notice that the stock is rebounding and converging towards a stable state.

Although managing the fishing catch may not be the most favored approach among fishermen, establishing a Marine Protected Area (MPA) could be a more viable solution for reviving depleted fishing stocks.

6. Effect of the surface size on MPA to destabilize the fish extinction equilibrium

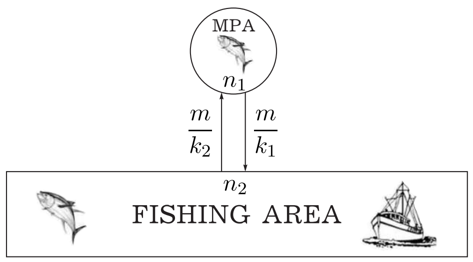

Fig. 11. Scheme of the system considered.

When the MPA has been implemented, the areas are structured in two distinct patches. Therefore, there are two principal components that make up the fish stock. A protected stock that stays around the MPA and a stock that which harvested by fishing fleets in the open sea (see Figure 11), with fish moving between the MPA and the open sea as well as fleets movements stays in the open sea. Assuming that fishes move at a fast time scale, and the fish growth occurs at a slow one. The fish movement rates m_i are supposed to be inversely proportional to the carrying capacity of the site and given by

$$m_i = \frac{m}{k_i}. \tag{11}$$

Where k_1, and k_2 are the carrying capacity of the protected and open sea areas respectively, and m represents a positive constant. $k = k_1 + k_2$ is the total fish carrying capacity.

Consequently, as per the formulation presented in equation (11), fish tend to stay in areas with higher carrying capacities, that is locations with plentiful resources. This assumption indicates that the distribution of fish among areas should conform to the ideal free distribution [11]. The model with MPA is described as follows:

$$\begin{cases} \dfrac{\mathrm{d}n_1}{\mathrm{d}\tau} = \dfrac{m}{k_2}n_2 - \dfrac{m}{k_1}n_1 + \varepsilon\left[rn_1\left(1 - \dfrac{n_1}{k_1}\right)\right], \\ \dfrac{\mathrm{d}n_2}{\mathrm{d}\tau} = \dfrac{m}{k_1}n_1 - \dfrac{m}{k_2}n_2 + \varepsilon\left[rn_2\left(1 - \dfrac{n_2}{k_2}\right) - \dfrac{qn_2}{(n_2 + D)}\dfrac{E}{1-s}\right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E\left(-c + p\dfrac{qn_2}{n_2 + D}\dfrac{1}{1-s}\right), \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha\left(\dfrac{A}{1+\beta p} - \dfrac{qn_2}{(n_2 + D)}\dfrac{E}{1-s}\right). \end{cases} \tag{12}$$

We denote the stock in the protected and fishing area by n_1 and n_2. $1-s$ represents the surface size of the unprotected zone, and s represent the surface size of the protected area. The last complete model can be aggregated. Firstly, we get the fast system by neglecting the slow part in the complete system, i.e. (we set $\varepsilon = 0$) in (12). Then, the fast equilibrium is given by

$$n_1^* = vn = \frac{k_1}{k}n,$$
$$n_2^* = (1-v)n = \frac{k_2}{k}n,$$
$$p^* = \frac{A\left[n - \frac{D}{1-v}\right] - nE\frac{q}{1-s}}{\beta nE\frac{q}{1-s}}.$$

We obtain the aggregated model by substituting the equilibrium for price and fish stock into (12), and by adding fish equations, and using slow time. The reduced system will take the form:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n\left[r\left(1-\dfrac{n}{k}\right) - \dfrac{QE}{n+L}\right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = -cE + \dfrac{A(n+L) - QnE}{\beta(n+L)}. \end{cases} \tag{13}$$

Where $n = n_1 + n_2$ represents the total fish biomass, E represents harvesting vessels, $Q = \frac{q}{1-s}$ is the global catch-ability parameter, and $L = \frac{D}{1-v}$. This presented system is the same as the previous system presented in the last section. It has a fish extinction equilibrium $\left(0, \frac{A}{c\beta}\right)$ which is unstable when the stated inequality is valid:

$$s < 1 - (1-v)w = s_{\max}, \tag{14}$$

with $w = \frac{qA}{cr\beta D}$. Equation (14) predicts an increase of the fish biomass inside the MPA that increases its carrying capacity, with the decreasing surface size of the fishing area, while the fish population in the fishing zone would decrease.

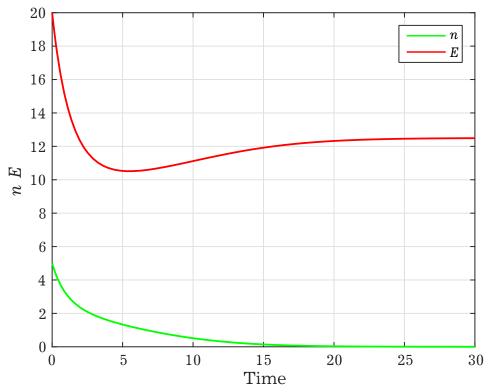

Fig. 12. Illustration of the case of a stable fish extinction equilibrium for $r = 0.5$, $k = 30$, $q = 1$, $D = 15$, $c = 0.4$, $A = 5$, $\beta = 1$, $n(0) = 5$, $E(0) = 20$.

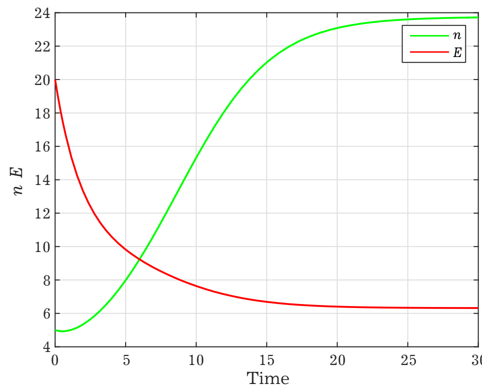

Fig. 13. Illustration of the case of implementation of MPA after a time $t = 5$, $u = 0.6$, $s = 0.2$.

Considering the case where we get a fish extinction equilibrium, Figure 12 without the installation of MPA. In Figure 13, we install MPA to our system after a time interval of t, we can notice that the fish stock is restoring and converging to a steady state of the system. In the next section, we identify the ideal surface zone and carrying capacity required for a MPA to achieve maximum capture at equilibrium.

7. MSY and the optimal size of a MPA

The maximum sustainable yield, denoted by "MSY" is the maximum rate of the resource catch at equilibrium state. The choice of the surface size of the MPA, has an important consequence on the harvesting rate. The total capture can be optimized, by optimizing the size of the surface of MPA. At an equilibrium state, if the harvesting rate exceeds MSY, it can result in the extinction of resources.

This section focuses on studding the case of stable equilibrium (n^*, E^*) so that we consider the case where $c > \frac{qA}{rD\beta}$ (i.e. $s < 1 - (1-v)w$).

The total catch-function at equilibrium is

$$Y^* = \frac{Qn^*E^*}{n^*+L} = rn^*\left(1 - \frac{n^*}{k}\right). \tag{15}$$

So, that

$$\frac{\mathrm{d}Y^*}{\mathrm{d}n^*} = r\left(1 - 2\frac{n^*}{k}\right).$$

Then $\frac{\mathrm{d}Y^*}{\mathrm{d}n^*} = 0$ means: $n^* = \frac{k}{2}$. Consequently, the value $n^* = \frac{k}{2}$, gives the maximum catch and it is presented in the following manner:

$$Y^*_{MSY} = \frac{rk}{4} \quad \text{at} \quad n^* = \frac{k}{2}. \tag{16}$$

Then, Y^* achieves a maximum equal to $\frac{rk}{4}$ for $n^* = \frac{k}{2}$ and it relates to the Maximum Sustainable Yield (MSY).

Substituting $n^* = \frac{k}{2}$ and $Y^*_{\text{MSY}} = \frac{rk}{4}$ in (15) gives s_{opt} as

$$s = s_{\text{opt}} = 1 - \frac{q}{rc\beta}\frac{(1-v)(4-rk)}{k(1-v)+2D}, \tag{17}$$

where s_{opt} represents the optimal surface size of MPA at equilibrium to prevent the Fish Extinction state.

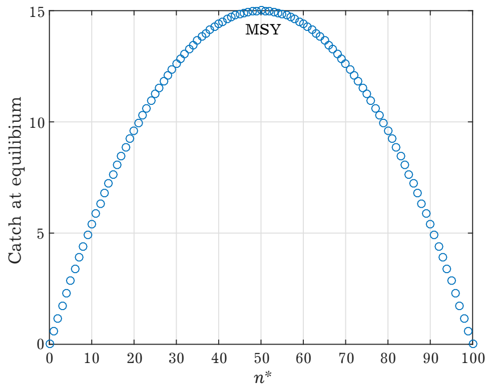

Fig. 14. Illustration of the catch at equilibrium with respect to n^*, with parameters set: $r = 0.6$, $k = 100$, $c = 0.6$, $q = 1$.

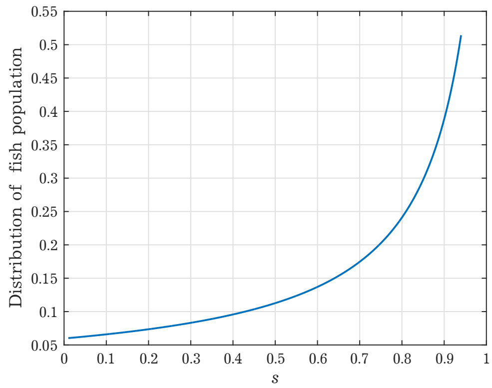

Fig. 15. Curve of the optimal surface size in terms of fish distribution.

In Figure 14, we can notice that the total catch shows a maximum for $n^* = \frac{k}{2}$, this is equivalent to the formula derived in equation (16), and Figure 15 shows that an increase of the fish biomass in the protected zone that increases its carrying capacity then it is surface size by consequence the fish biomass in MPA would increase.

8. Conclusion and perspectives

A mathematical model presented that includes two main aspects, the use of a non-linear harvesting function, and economical aspects corresponding to the price variation, according to the gap between the demand which was supposed to be a nonlinear; monotonous decreasing function, and the supply which is the capture. Moreover, we have assumed that the price occurs at a faster time scale and the fishery dynamics and the population growth follow a slow one. Under these assumptions, we obtain an aggregated model with 2 ordinary differential equations. The use of a nonlinear demand function and a nonlinear harvesting function has a significant impact on the ecosystem of the fishery.

The study of the model reveals that, according to parameter values, one up to two non-trivial positive equilibrium points can exist. We have established distinct criteria for the parameters that result in various cases. To summarise, the study of the reduced system shows that three-main cases can occur:

1) The existence of a unique positive equilibrium that can be stable if the cost per unit of fishing effort increased, this lead to a decrease in fishing activities, and as a consequence, the fish biomass recovers.
2) A noticeable case corresponding to fish extinction. When the fish population becomes close to extinction, the price in the market remains very expensive so the income remains good for boat owners, that is why the harvesting vessels continue to harvest the stock until depleted due to the large net benefit.
3) The coexistence of two interior positive equilibria. An equilibrium with a low biomass level with a risk of extinction which is a saddle point, and another one with a large biomass level, far from extinction but the fishery can only handle a small amount of fishing pressure, which is locally asymptotically stable. In this case, the FEE remains stable. As a result, there exists a separatrix between both stable equilibria. That means; either the fish biomass will become extinct with a rise in price, or a stable sustainable state will be reached.

By using a non-linear demand function an interesting case occurred corresponding to fish extinction. To avoid this catastrophic case, it is essential to identify strategies that prevent fish extinction case. Firstly, we add a control parameter to the system, by controlling the fishing effort. We find that it is essential to keep the control parameter below a specific threshold value to avoid the fish extinction case. Secondly, another crucial aspect we explored, is determining the ideal size and capacity of marine protected areas (MPAs) to prevent fish species from extinction while also maximizing catch equilibrium. We studied a fishery mathematical model with a protected area (MPA) and an unprotected zone (i.e. fishing zone). By utilizing this model, we can observe the impact of implementing MPAs on depleted fishing stocks. Through this model, we were able to identify the most effective surface area and carrying capacity of an MPA to achieve maximum catch at equilibrium.

As a perspective, Artificial intelligence (AI) and neuroscience can be applied in fisheries in various ways to improve the efficiency and sustainability of fishing practices. We would like to look for the use of neuroscience to analyze the behavior of fish in their natural habitats. This information can be used to develop better fishing areas. It is also crucial to study the influence of the multi-patches on the dynamics of the fishery and the conversion of a depleted fishery represented by an over-exploitation to a sustainable state. We can modify the fishing effort equation to include a time-dependent control function, representing the proportion of fishing profits invested, and also to use stock-effort-dependent prices and coasts, which would be more interesting.

Appendix

A1. Calculation of the fast equilibria

At the fast time: $D(p) - \frac{qnE}{n+D} = 0$. It follows that $\frac{A}{1+\beta p} = \frac{qnE}{n+D}$, and $p^* = \frac{A(n+D)-qnE}{\beta qnE}$.

A2. Derivation of the reduced models

Replacing the fast price equilibrium p^*, then

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = \varepsilon n\left[r\left(1-\frac{n}{k}\right) - \frac{qE}{n+D}\right],$$
$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E\left(-c + \frac{qn}{n+D}p^*\right).$$

Then, the total fish stock equation is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = n\left[r\left(1-\frac{n}{k}\right) - \frac{qE}{n+D}\right].$$

Respectively, the total fishing efforts equation is

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \left(-c + p^*\frac{qE}{n+D}\right)E,$$

then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -cE + \frac{A(n+D) - qnE}{\beta(n+D)}.$$

A3. Stability analysis of equilibria

Consider the positive equilibrium (n^*, E^*) of system (4) that solve:

$$\begin{cases} E(n) = \dfrac{r}{qk}(k-n)(n+D), \\ E(n) = E = \dfrac{A(n+D)}{c\beta(n+D)+qn}. \end{cases} \tag{18}$$

Because

$$F(n) = n^2(rc\beta + rq) + n(rc\beta D - rc\beta k - rqk) + (Aqk - rc\beta kD),$$

then

$$F'(n) = r(\beta c + q)\left[2n - k + \frac{\beta cD}{\beta c + q}\right]$$

and $F'(n) = 0$ if $n = \frac{1}{2}(k - \frac{\beta cD}{\beta c+q})$.

The number of equilibria with positive values of the system (4) is given by the solutions of $F(n) = 0$. We get:

► Case 1: If $\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) > 0$,

n	0		$\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)$		$+\infty$
$F'(n)$		$-$	0	$+$	
$F(n)$	$-r\beta cD + Aq$	↘	$F\left(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\right)$	↗	$+\infty$

- If $Aq - r\beta cD < 0$, so equation $F(n) = 0$ have a single positive solution.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) < 0$, then equation $F(n) = 0$ have two positive solution (n_1^*, E_1^*) and (n_2^*, E_2^*) verifying $n_1^* < \frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) < n_2^*$.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) > 0$, then equation $F(n) = 0$ do not have positive solutions.

► Case 2: If $\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) < 0$,

n	0		$\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)$		$+\infty$
$F'(n)$			$+$		
$F(n)$	$-r\beta cD + Aq$		↗		$+\infty$

- If $Aq - r\beta cD < 0$, so equation $F(n) = 0$ have a single positive solution.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) < 0$, then equation $F(n) = 0$ do not have positive solutions.

A4. Local stability analysis of the non-trivial equilibria

At (n^*, E^*), the Jacobian matrix of system (4) is

$$Jac_{(n^*,E^*)} = \begin{pmatrix} \frac{rn^*}{k(n^*+D)}[(k-D)-2n^*] & \frac{-qn^*}{n^*+D} \\ -\frac{Dr(1-\frac{n^*}{k})}{\beta(n^*+D)} & -\frac{\beta c(n^*+D)+qn^*}{\beta(n^*+D)}) \end{pmatrix}.$$

The trace is given by

$$\begin{aligned} \operatorname{tr} Jac^* &= \frac{rn^*}{k(n^*+D)}[(k-D)-2n^*] - \frac{\beta c(n^*+D)+qn^*}{\beta(n^*+D)} \\ &= \frac{-1}{k(n^*+D)\beta}\Big[2r\beta n^{*2} + n^*\big[r\beta(D-k)+k(\beta c+q)\big] + \beta ckD\Big] \\ &\leqslant -\frac{F'(n^*)\,n^*}{k(n^*+D)(\beta c+q)} - \left(c + \frac{qn^*}{\beta(n^*+D)}\right). \end{aligned}$$

So, if $F'(n^*) > 0$, $\operatorname{tr} Jac^*$ become negative.

The determinant is given by

$$\det Jac^* = \frac{n^*}{k\beta(n^*+D)}\Big[-2n^{*2}(\beta c+q) + n^*\big[k(\beta c+q) - 2D(\beta c+q) - \beta cD\big] + \big[kD(\beta c+q) - \beta cD^2\big]\Big].$$

Given that: $F(n^*) = rn^{*2}(\beta c+q) + n^* r(\beta c+q)\big[\frac{\beta cD}{(\beta c+q)} - k\big] + k(Aq - r\beta cD)$,

$$\det Jac^* = \frac{n^*}{k\beta(n^*+D)}F'(n^*).$$

We can see that $\det Jac^*$ and $F'(n^*)$ have the same sign. It is possible to distinguish two main cases:

► Case 1: If $\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) > 0$.

- If $Aq - r\beta cD < 0$, then equation $F(n) = 0$ have a single positive solution (n_2^*, E_2^*), which sign $\big(\det Jac_{(n_2^*,E_2^*)}\big) = \operatorname{sign}\big(F'(n^*)\big) > 0$, and $Trac_{(n^*,E^*)} < 0$, then it is stable.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) < 0$, then equation $F(n) = 0$ have two positive solution (n_1^*, E_1^*) and (n_2^*, E_2^*) verifying $n_1^* < \frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) < n_2^*$. Since sign $\big(\det Jac_{(n^*,E^*)}\big) = \operatorname{sign}\big(F'(n^*)\big)$. Therefore, we can state the following results:
  1) The determinant $\det Jac_{(n_1^*,E_1^*)}$ is negative, the positive equilibrium (n_1^*, E_1^*) is then a saddle point.
  2) The determinant $\det Jac_{(n_2^*,E_2^*)}$ is positive, because sign $\big(\det Jac_{(n_2^*,E_2^*)}\big) = \operatorname{sign}\big(F'(n_1^*)\big) > 0$, and $Trac_{(n_2^*,E_2^*)} < 0$, then the positive equilibrium (n_2^*, E_2^*) is stable.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) > 0$, then equation $F(n) = 0$ do not have positive solutions.

► Case 2: If $\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big) \leqslant 0$.

- If $Aq - r\beta cD < 0$, so equation $F(n) = 0$ have a single positive solution (n_2^*, E_2^*), which sign $\big(\det Jac_{(n_2^*,E_2^*)}\big) = \operatorname{sign}\big(F'(n_2^*)\big) > 0$, and $Trac_{(n_2^*,E_2^*)} < 0$ then it is stable.
- If $Aq - r\beta cD > 0$ and $F\big(\frac{1}{2}\big(k - \frac{\beta cD}{\beta c+q}\big)\big) < 0$, then equation $F(n) = 0$ do not have positive solutions.

[1] Clark C. W. Mathematical Bioeconomics: The optimal management of renewable resources. Wiley, New York (1990).

[2] Mchich R., Auger P., El Abdlaoui A. Méthode d'agrégation des variables appliquée à la dynamique des populations. Revue Africaine de Recherche en Informatique et Mathématiques Appliquées. **5**, 26–32 (2005).

[3] Mchich R., Auger P., Brochier T., Brehmer P. Interactions Between the Cross-Shore Structure of Small Pelagic Fish Population, Offshore Industrial Fisheries and Near Shore Artisanal Fisheries: A Mathematical Approach. Acta Biotheoretica. **64**, 479–493 (2016).

[4] Bazykin A. D. Nonlinear Dynamics of Interacting Populations. World Scientific, Singapore (1998).

[5] Edelstein-Keshet L. Mathematical Models in Biology. Random House, New York (1998).

[6] Barbier E. B., Strand I., Sathirathai S. Do open access conditions affect the valuation of an externality? Estimating the welfare effects of mangrove-fishery linkages. Environmental and Resource Economics. **21**, 343–367 (2002).

[7] El Hakki I., Mchich R., Bergam A., Charouki N., El Harrak A. Effect of a nonlinear demand function on the dynamics of a fishery. Mathematical Modeling and Computing. **10** (4), 1143–1153 (2023).

[8] Auger P., Mchich R., Raïssi N., Kooi B. Effects of market price on the dynamics of a spatial fishery model: Over-exploited fishery/traditional fishery. Ecological Complexity. **7** (1), 13–20 (2010).

[9] Brochier T., Auger P., Thiao D., Bah A., Ly S., Nguyen-Huu T., Brehmer P. Can overexploited fisheries recover by self-organization? Reallocation of fishing effort as an emergent form of governance. Marine Policy. **95**, 46–56 (2018).

[10] Ly S., Balde M., Mansal F., Nguyen-Huu T., Auger P. A Model of a Multi-Site Fishery with Variable Price: from Over-Exploitation to Sustainable Fisheries. Mathematical Modelling of Natural Phenomena. **8**, 130–142 (2014).

[11] Mchich R., Auger P., Bravo de la Parra R., Raïssi N. Dynamics of a fishery on two fishing zones with fish stock dependent migrations: aggregation and control. Ecological Modelling. **158** (1–2), 51–62 (2002).

[12] Schaefer M. B. Some considerations of population dynamics and economics in relation to the management of the commercial marine fisheries. Journal of the Fisheries Board of Canada. **14**, 669–681 (1957).

[13] Ly S., Auger P., Balde M. A bioeconomic model of a multi-site fishery with nonlinear demand function: number of sites optimizing the total catch. Acta Biotheoretica. **62** (3), 371–384 (2014).

[14] Moussaoui A., Auger P. A bioeconomic model of a fishery with saturated catch and variable price: Stabilizing effect of marine reserves on fishery dynamics. Ecological Complexity. **45**, 100906 (2021).

Динаміка рибного промислу з нелінійним виловом: контроль, коливання ціни та MSY

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У цій статті будується та аналізується нова математична модель рибальства, що описує часову еволюцію рибних запасів, які виловлює рибальський флот, що описується його промисловим зусиллям. Вважається, що ціна, яка визначається різницею між попитом і пропозицією, змінюється з часом. Для функції збирання використовуємо функцію Холлінга II. З іншого боку, розглядається два різні часові масштаби: швидкий для зміни ціни та повільний для рибного запасу та зміни риболовного зусилля. Використовується метод "агрегування змінних", щоб отримати агреговану модель, яка керує біомасою риби та рибальським зусиллям у повільному часі. Аналізуючи цю скорочену модель, за певних умов доводиться, що можуть виникнути три цікаві рівноваги. Крім того, показано як можна керувати моделлю, щоб уникнути небажаних ситуацій і досягти стійкої рівноваги. Іншим цікавим аспектом, наведеним у цій статті, є можливість впровадження морських заповідних територій (МЗТ). Показано, як МЗТ дозволяє нам внести значний внесок у відновлення виснажених популяцій риб. Це досягається шляхом порушення стану рівноваги "вимирання риби" та встановлення стійкого стану.

Ключові слова: *рибальська модель; функція Холлінга II; різна ціна; агрегування змінних; рівноваги; стійкість; контроль; морська заповідна зона; максимальний стійкий вихід.*